The Mathematics Teacher

MARCH 1958

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MINA REES

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ROBERT M. DURKEE

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The impact of the computer[1]

MINA REES, *Hunter College, New York, New York.*

Today, computers are used to solve problems which were "out of reach" just a few years ago. Students in your high school classes may have a role to play in the future development of these "brains" if they have a sound mathematical background.

MY THEME IS THE COMPUTER, and its impact upon our changing world. Let me for a moment consider my subject in the perspective of the onward march of mathematical research. Mathematics has been called both the queen and the handmaiden of the sciences; and it is as handmaiden that mathematics usually appears in its association with computers. The computer does not provide the imaginative insights, akin to poetry, which characterize the most significant research in pure mathematics, although it sometimes provides the material on which such insights are based. The computer is a servant, strictly obedient and highly versatile, but it must be instructed, as Claude Shannon has said, in words of one microsyllable.

The first commercially built large-scale computer in the United States was put into operation in time for limited application in the preparation of the 1950 census. Since that time, there have been great strides in the development of these "giant brains." Their use has expanded to include applications not only in science and engineering, and in business and industry, but also in many directions not dreamed of by the early users. I shall try in this short paper to give you only a glimpse of some of these directions.

[1] This article is based on an address given at a "Conference on Science in Perspective" at Randolph-Macon Woman's College, November 14, 1957.

What is the impact of the computer? What kinds of problems with solutions inaccessible until now have come within reach with its advent? There are several categories. In the first case, the *mathematical nature* of the solution of the problem was fully understood before the coming of the computer, but the solution could not be found. Here a mathematical model had been formulated, but could not be tested against the real world because numerical answers were not available. These could not be found either because the time required to obtain a solution was prohibitive, or else because the method to be used had not been developed. In this category is the problem of orbit computation when the need for speed is great. If we are merely interested in computing the path of a heavenly body as it moves on its leisurely way through space, then the older punch-card equipment can be made to do the job, not so well as the new equipment but adequately. But if we want to predict the position of Sputnik II as it races around the world, then powerful and fast modern equipment is needed. In this category, also, is the problem of flood control. Within the past few years the United States Army Corps of Engineers has become interested in numerical flood prediction and river regulation, and a study has been completed for them by a group

at New York University headed by Professor J. J. Stoker. This group has studied floods in the Upper Ohio River from Wheeling to Cincinnati, at the junction of the Ohio and the Mississippi Rivers, and in the Kentucky Reservoir. The group has demonstrated that, at least for the junction of the Ohio and Mississippi and for the Kentucky Reservoir, and probably whenever the engineering data are good enough, machine computation will solve the prediction and control problems more cheaply and more satisfactorily than the older engineering methods. The Corps of Engineers has adopted the NYU method as a basis for settling design questions for a reservoir to be built in the Chattahoochee River in Alabama. Of this flood control problem I shall say more later.

Other engineering problems that have been attacked numerically because of the availability of the computer are too numerous even to suggest. Among mathematical problems it may be of interest to mention the study of a Galois field using a computer to determine primitive roots.

A second category includes the problems that simply were not formulated for solution before the coming of computers. Numerical weather forecasting is an instance of this. The classical weather equations describing the motion of the atmosphere are well known, and are related to those used in the flood control problem; but an adequate picture of the weather cannot be given by a model as simplified as that used in the study of floods. To handle this problem a model had to be devised which would, on the one hand, be capable of numerical solution and, on the other hand, check against the realities of the weather. Pioneering work of the late John von Neumann in collaboration with J. G. Charney evolved a method for attacking this problem. Their method was not only sufficiently accurate to produce reliable predictions, it also was sufficiently fast to predict the weather before it happened. This was one of the early bottlenecks—that it took more than twenty-four hours to find the solutions of the weather equations necessary to predict the next day's weather. Now we have a Joint Weather Prediction Unit, a combined activity of the United States Weather Bureau, Air Force, and Navy, equipped with a high-speed computer and turning out twelve-, twenty-four-, and thirty-six-hour prognostic maps daily.

A third category of problems receiving a new look is the concern of business and industry. At the beginning of 1955, the editors of the magazine *Management Methods* published a report on "What Management Should Know about Electronics for the Office," in which they said:

> The time has come for the businessman to drop the future tense as far as the Automatic Office is concerned. Managements who fail to grasp this new condition—and its implications—may find themselves hopelessly outdistanced by alert competitors before they have time to act.[2]

Since the writing of this article, there has been a good deal of dispute among businessmen as to whether the very high speed of the computer is needed for business uses. On this point the October, 1957, issue of *Computers and Automation* carried an article entitled "Are Automatic Computer Speeds Faster than Business Needs?" The author, Ned Chapin, concludes that the machines are not too fast, but that often it will require a basic redesign of the business problem that the machine is asked to solve to get full value from the machine.

This article concludes:

> Since an automatic computer can perform only data processing that could be performed by other means, the business acceptance and superiority of an automatic computer must lie in the areas of cost and speed. . . . Without speed, the use of an automatic computer can rarely be justified in business on a cost basis . . . in general, the higher the speed, the lower is the per unit cost of data processing accomplished.[3]

[2] *Management Methods* (January, 1955), pp. 10–13.
[3] *Computers and Automation*, VI, No. 10 (October, 1957), 12–17.

However much the need for high speeds is disputed, the fact is that industry is installing computers at a bewildering rate. Computers are used by insurance companies to compute the benefits on a policy and to bill customers. They are used by mail order houses and other firms to maintain inventory, and they are used by airlines to monitor reservations. Banks have research in progress to develop computers for the automatic handling of checks; and the telephone company is developing new computer techniques to determine charges on privately leased long-distance telephone lines. These are just a few of the business uses of computers. The magazine *Computers and Automation* estimates the market for computers, for all purposes, as two billion dollars per year by 1965.[4]

A fourth category of problems is represented by the mechanical translation of foreign tongues. Problems in this fourth category do rely on speed for their successful solution as do all the problems we have mentioned; but in this group a substantial and extended effort is required by specialists in fields other than mathematics and engineering to reduce the problem to a form where the machine can take over. There is widespread activity among linguists, not only in this country but in many other countries as well, all of it aimed at understanding more clearly the essential elements involved in the process of translating. The success of many of these efforts is essential to a satisfactory solution of the mechanical translation problem.

In estimating the impact of the development of computers, we must recognize that the coming of computers has stirred the imagination of workers in many fields, so that a stream of ideas and questions has been generated, occasionally by engineers and mathematicians, but more often by men with no special knowledge of computers but with problems to solve. For them the computer has brought a gleam to the eye. For example, the Patent Office has asked the Bureau of Standards to develop a patent-searching machine. A patent search is both exhaustive and exhausting. If the operation of search can be described in minute detail, and if the implied engineering problems can be solved, then there is hope of simplifying the enormously complex operation of the Patent Office. It may be of interest to mention one aspect of this study for the Patent Office that was recently reported—a study of the frequency of occurrence of various types of sentence structure in scientific articles.[5] This information is of interest not only for the mechanized searching of patents but also for machine translation of languages. The Post Office, too, has asked the Bureau of Standards for assistance in its effort to mechanize the sorting of mail.

A final category of problems consists of those that rely on the completion of computations in what we call "real time," computations such as those that will enable the tower of an air terminal to control automatically, without the intervention of the pilot, the paths and times of arrival of incoming planes, or will provide automatic interception of incoming missiles. In the recent discussion of trips to the moon, touched off by the successful launching of the first Russian earth satellite, one writer said:

> After a few weeks in the sky the pilot will be ready to return to the surface of the earth. Now an inordinately complex electronic computer will take over to bring the rocket plane down. The speed of the rocket plane will be so high and the reaction time of the pilot so slow that only an electronic "brain" will be capable of performing this exacting task.[6]

This is an example of the "real-time" use of a computer. The course of our

[4] Ed Burnett and Leland Hewitt, "The Market for Computers in Banking: Report No. 2," *Computers and Automation*, VI, No. 9, 6, 7, 30.

[5] National Bureau of Standards, "Syntax Patterns in English by Electronic Computer," *Computers and Automation*, VI, No. 7 (July, 1957), 15–17, 32.

[6] I. M. Levitt, "Now the Space Age Opens," *New York Times Magazine*, October 18, 1957, pp. 19, 82–84.

rocket plane must be computed as fast as it is traversed so that the computer may make its decision soon enough to instruct our spaceship to change its course. Great strides have been made in the past few years in achieving real-time computation, both through the speedy processing of data, an engineering achievement, and through advances in information theory and related mathematical questions.

This brief sketch may serve to give you a broad view of many of the facets of current computer applications. To add depth to the picture, I shall return to two of the most interesting and most representative of the problems I have mentioned—flood control, and mechanical translation—and discuss them in greater detail.

In the flood control problem, mathematicians have moved forward with the constant help of engineers. What is the nature of the problem?[7]

The Corps of Engineers of the United States Army is responsible for the prediction of the flow in the Ohio and Mississippi Rivers as well as in other channels, i.e., for estimating, in advance, the height the water will reach under flood conditions; and for the design and construction of reservoirs that will contain the floods and prevent disasters.

How has the Corps of Engineers discharged this obligation? Like its counterpart organizations in other parts of the world, the Corps of Engineers has usually studied the flow in such channels by constructing scale models of the rivers and reservoirs that are under study. These models occupy acres of ground. Their construction requires detailed knowledge of the length and breadth of the river, the changing cross section, and the inflow from tributaries plus the local inflow from the main channel. The roughness of the river bed that influences the flow in an essential way, giving the flow its turbulent character, cannot be scaled. This roughness, together with gravity, is the biggest force conditioning the flow of the stream. In the engineering model this roughness is approximated by screwing knobs onto the bed of the river model, and, after varying the size and position of these knobs, testing the result until the modeled flow approximates closely enough the real flow. The cost of a single such model is comparable to the cost of an electronic computer. Moreover, if a computer *can* handle this problem, then, after the basic data for a particular river or reservoir have been prepared, and the original programing of the computer has been completed, simple changes will enable us to solve all sorts of related problems quickly and inexpensively, for example, the effect on a flood wave of closing off a tributary by operating a dam, or the effect of changing the schedules of power dams at the ends of large reservoirs.

It was thus natural for the Corps of Engineers to ask several years ago whether it would be possible to obtain, mathematically, results comparable to engineering results.

The answer to the question asked by the Corps of Engineers has now been given in the affirmative. The NYU group has shown that, for long rivers, where the mathematical description of the flow can be given by sufficiently simple equations, an electronic computer can give the needed predictions with satisfactory accuracy.

But the equations describing this type of flow have been known for a long time. Why must we have a computer to solve them?

The important thing to observe is that the equations that describe the flow cannot be solved by a formula like the ones used to solve equations in elementary algebra, for example. The present equations are nonlinear partial differential equations. If we *could* find formulas for the

[7] The following discussion is based on an unpublished paper by J. J. Stoker, and a report by E. Isaacson, J. J. Stoker, and A. Troesch, "Numerical Solution of Flood Prediction and River Regulation Problems," prepared under the sponsorship of the United States Army Corps of Engineers, Ohio River Division, October, 1956.

solutions, the formulas would give the velocity of the water v, and its height above the sea level H, in terms of the distance down the river x and the elapsed time t. But mathematicians are not able to write down an expression for H and v in terms of x and t. In the flood-prediction problem this means that instead of having the height and velocity of the water described at every point of the river, we can find their values only for selected numerical values of x and t. In the case we are studying, it was shown that sufficient accuracy could be obtained by finding solutions at ten-mile intervals and spaced nine minutes apart.

In the actual process of solution the differential equations are approximated by multitudes of linear algebraic equations—so many that, although mathematicians know how to solve them theoretically, it would be impossible to complete the numerical solution in a reasonable time without the aid of machines. Using an early model of the UNIVAC, the NYU group produced a flood prediction for the Ohio River, covering $13\frac{1}{2}$ days, in about $6\frac{3}{4}$ hours of computing time. With faster, newer equipment the computation should take less than one-half hour.

Before the equations can be solved, however, the coefficients that characterize a particular river must be found with sufficient accuracy so that the resulting equations will give a picture of the flow, and can therefore be used for predictions. Here we need the cross-section area and the breadth of the river at ten-mile intervals, the roughness and slope of the river bed, and the inflow of water into the river from tributaries and local drainage areas. In addition to these data, initial and boundary conditions must be known. We need, for example, the height and velocity of the water at the time when the study begins. The collecting and processing of the data are laborious and crucial. For the study of a 375-mile stretch of the Upper Ohio River, 1100 constants were needed.

This is not a problem to be solved by a mathematician sitting in his ivory tower. Its solution required the fullest co-operation between mathematicians and engineers. This problem and its solution represent the kind of effort that is going on in research and development across the country. In this particular problem, the mathematical analysis showed something not known from engineering studies—that upstream waves are sometimes of great importance. These upstream waves occur in a region that can be determined mathematically from the differential equations. In a long river this upstream effect is usually small. In such a case the study of the flow by means of engineering models usually gives satisfactory results. However, if the tributary inflow is so large as to cause large back-water effects, like those at the junction of the Ohio and Mississippi, the engineering method runs into trouble. And in the Kentucky Dam where the propagation upstream due to reflection from the dam is a factor of major importance (here a wavelet propagates with great speed—it covers the 184-mile length of the dam in approximately $\frac{1}{4}$ day), the engineering method has never been satisfactory.

I hope this discussion has given some idea of the scope of the contribution the mathematician and the computer can make to a large engineering problem. In the aircraft, communications, electrical, and many other industries, as well as in many activities of government, this pattern of assistance could be repeated many times.

Let me move on now to the translation problem in which the approach is somewhat different. As far as is known in the Western world, it was an American mathematician, Warren Weaver, in conversation with an English physicist, A. D. Booth, who first seriously considered the possibility that mechanical translation from one language to another might be technically feasible. However, the pursuit of this difficult objective has involved great numbers of linguists as well as

natural scientists and engineers in many countries of the world. The analysis of language with an eye on the capabilities of the computer lies at the heart of the problem.

Some of the difficulties of machine translation seem similar to those faced when any problem is to be solved by a machine. The original text must be fed into the machine and transformed into symbols the machine can understand; the machine must then operate on the data; and finally the results must be transformed into the written or oral form of the desired language. In the translation problem there is some hope that engineering methods will be developed that will enable the computer to read a printed text without the intervention of a human being. This engineering research would benefit greatly from the developmental work that is being sponsored by banks and is aimed at acquiring automatic check-handling equipment. One American bank has estimated that it could save over two million dollars a year if machines could replace hand labor in this operation. At least eight research laboratories in the United States are working on the problem, and have had some success.

The central problem in machine translation is the way in which the machine operates on the data. The most widely publicized effort directed at using an existing computer to perform a complete translation was carried on in January, 1954. At that time the IBM 701 was programed to use linguistic results developed at the Georgetown Institute of Languages and Linguistics to translate a number of Russian sentences. The machine used a 250-word vocabulary and six Russian grammatical rules, and it automatically changed the word order so that the output was expressed in acceptable English. It was clear from this experiment that even with adequate linguistic analysis an enormously larger memory would be needed to handle translations that had not been artificially delimited. Since that time the size of computer memories has been vastly expanded. There is now reason to hope that by limiting the vocabulary to specialized fields (mathematics and brain surgery are two such fields that have been under study) and by taking advantage of the findings of extensive and detailed linguistic research, we should be able to get useful translations within a few years. Research now in progress is concerned with such problems as the stripping off of inflectional endings (and the derivation of grammatical meanings from these endings); the recognition that problems of multiple meaning derive largely from the study of a word in isolation and the demonstration that the maximum information about the meaning of a word comes from the first one or two words on either side of it; the handling of irregular words and of idioms; the study of the characteristic word order of various languages and the derivation of procedures for machine coding to transform this word order into one acceptable in the target language; compilation of specialized glossaries of meaning-bearing words for specialized fields; and the recognition that high frequency words—articles, prepositions, conjunctions, auxiliary verbs—provide the grammatical structure in which the nouns, verbs, adjectives, and adverbs (the meaning-bearing words) are held. It is of interest to consider reports of some of the Russian work in this field.

Two of the articles published by the Russian Academy of Sciences in 1956 deal with machine translation from English to Russian and describe methods used in programing the BESM (Russian) computer for this purpose. One of these articles is reported to be

> A short description of the methods used in programming the BESM computer to translate from English to Russian. A vocabulary of 952 English words was used, augmented by a routine for separating affixes s, 's, ing, ed, er, est, e, th. Special routines were used with 121 of the English words for distinguishing multiple meanings on the basis of the preceding or following

words. The program analyzes the English sentence in a similar manner, changes the word order and synthesizes the Russian sentence, adding inflections of stems.[8]

And an article which appeared in the September-October, 1956, issue of a Russian journal (*Voprosy Yazykozwania*) dealing with linguistics, discusses

> . . . the principles of formulating a dictionary and a "grammar" (a system of special rules) for mechanical translation of a scientific text in the field of mathematics from French to Russian, and describes the translation process itself.
>
> A translation requires the following data: (1) a stem dictionary; (2) an idiom dictionary; (3) preposition translation tables; (4) rules for distinguishing homographs; (5) affix tables (in French and Russian); (6) groups of analyzing rules; (7) groups of synthesizing rules. These data are stored in the "memory" of the machine as needed.[9]

This Russian journal has initiated a series of articles devoted to mechanical translation, the first three of which appeared in the September-October, 1956, issue. Information about this work as well as other work around the world on this subject is reported in our own magazine, *Mechanical Translation* (MT for short, whether as a sly commentary on the subject or not, it is hard to know). There is active work in this field in England, France, Italy, Germany, and the Scandinavian countries, as well as in Russia and the United States.

Although there are no expectations that sensitive translations of great works of literature can be achieved by mechanical means, there is the hope that useful translations of technical articles in limited fields can be produced, and that, relatively soon, we shall be able at least to get machine translations which inform us reliably what articles merit the careful efforts of fully competent, and equally rare, translators who control not only the languages but the subject matter.

Through the discussion of these two examples, flood control and mechanical translation, I have tried to give an insight into two important kinds of problems for which computers are being used. In the first case, the mathematician makes a contribution to a highly sophisticated engineering problem, often using newly developed techniques of numerical analysis. In the second case, the problem is being newly explored, and the mathematician is often only a minor member of the team which is studying it and formulating it for the computer.

In the future, problems not yet dreamed of will be studied in this way, and we should see highly imaginative approaches to new situations that will make the computer an increasingly useful servant, as men open up unexplored paths leading to the far horizons. The young people now in our colleges and universities, and indeed those in our secondary schools, will have an opportunity to play a role in this development. Already we are being bombarded with the demand for students with mathematical training to work with computers. Though courses in computer programing and numerical analysis are being introduced into some of our colleges and universities, and can be very helpful, the more important need is for sound training in mathematics combined with experience in some other field (usually the sciences or the social sciences) and interest in working with other people. With such training our young men and women can find exciting careers in the world of automation if they are willing to embark in new directions with a spirit of high intellectual adventure.

[8] *Mechanical Translation*, III, No. 1, 29.
[9] *Ibid.*, No. 3, 91.

How much—or—how many?

ROBERT M. DURKEE, *Mississippi Southern College, Hattiesburg, Mississippi.*

An elementary discussion of digital and analogy computers which your junior and senior students might enjoy.

HAVE YOU NOTICED in the last few years how more and more applications are being found for high-speed computers? We see pictures of complex computers that can do stupendous computations in an incredibly short time. On TV, a man and woman are picked by a computer from thousands of names as being most ideally suited for each other. Reading the paper, we note how some young boy has constructed a computer which will tell you what plays to make in tick tack toe so that you cannot lose. In a detective story, the hero uses the computer to find the name of an intended murder victim.

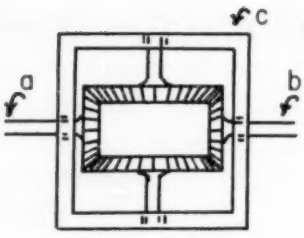
In the midst of all these glorious accomplishments, we are suddenly sobered by the want ads. We read, "Wanted: Computist, college graduate, with major or minor in math," or maybe a big advertisement by some well-known computer company: "We need mathematicians—to do research in computers—to process data for computers—to learn to repair computers—to teach computer theory." As mathematics teachers, we realize the growing need for many more young people to enter the "computer" field. Many of these young people will have their start in our classes.

Although progressive schools have already incorporated some instruction on computers in their curricula, most of us are having to find out what we can about computers on our own. There are many articles and numerous books on computers, but they are highly specialized. At present there are few books that describe computers and computer theory specifically for the secondary mathematics teacher. This article therefore is an attempt to describe some of the basic operations and applications of computers. We hope to explain these in a way that will show how high school or college students can learn more about computers.

When we speak of computers, we normally think of the latest high-speed machines. Actually, a computer is any device that will help us determine a quantity or an amount. The primitive people of long ago used to indicate "how many" with their fingers; thus we may consider fingers—or digits—the first computers. Not only do we think of digits as computers in general, but also as a certain type of computer, namely, "digital computer." A digital computer is the type of computer in which everything is in digits —whole numbers; the quantities are discrete, and there are no in-between values. Examples of digital computers are adding machines, desk calculators, cash registers, mechanical bookkeeping machines, telephone dials, and punched-card systems.

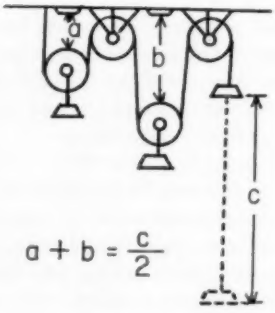
We can consider that the Greeks were basically responsible for starting another type of computer. They used compasses which did not indicate "how many" units in a given line segment. Their straightedges likewise were not marked off in units. They were interested in "how

much" without being restricted by numerical calculations. The slide rule, a comparatively recent invention, is an example of a computer which is used to compute "how much." This type of computer uses continuous values. Since there is no decisive break between one value and the succeeding value, we refer to these as "analog" computers. The word "analog" is not, as one might believe, a combination of some form of "log" (logarithmic) computing; instead it is merely a shortening of the word "analogy." Notice that the analogy in the slide rule is that numbers are represented by a physical quantity—the length that one calibrated line is moved along another calibrated line. Some common examples of analog computers are weight scales, rulers, pressure gauges, and most meters—electric meters, gas meters, speedometers, parking meters, hydrometers, thermometers, and many others. Note that in all these cases we may use discrete numbers in describing the amounts, whereas actually the values are continuous.



$a + b = 2c$

Figure 3



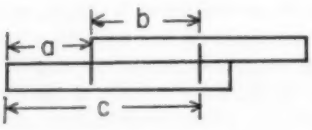
$a + b = \frac{c}{2}$

Figure 4

Analog computations

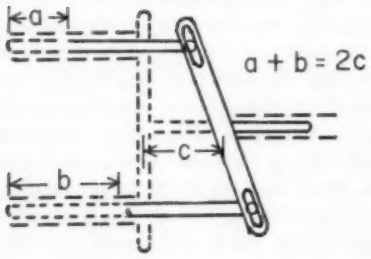
Many analog computers will do more than merely represent a physical or electrical quantity numerically. There are analog computers that will add, subtract, multiply, divide, and do other mathematical computations. Since computations done by an analog computer are normally more direct than those done by a digital computer, let us look at some of the ways that we have of figuring "how much" by means of analog computers. The accompanying figures show a few ways in which addition can be done mechanically.

Figure 1



$a + b = c$

Figure 2



There are other ways in which addition can be done mechanically, but these diagrams are sufficient for our purposes. The sliding-scale addition (Fig. 1) can be illustrated simply with two rulers. By sliding the top ruler along until its left index is above the first number to be added, the answer is found directly below the second number of the addition. In the linkage addition (Fig. 2), the amount that the middle bar is moved to the right de-

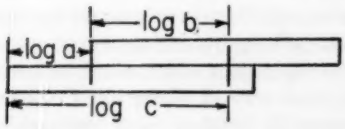
pends upon the other two. Although the total movement of a plus b is twice the amount that c actually moves, this presents no problem since the c scale can be calibrated accordingly. This is also true of the calibration of the c indicators in the differential and pulley adders.

The differential gear (Fig. 3) is convenient to use where the a and b values are in terms of the amount of rotation. You can see that if a is held stationary while b makes a complete revolution in the direction indicated, the housing c will go halfway around in the direction indicated. This action would be similar if we held b and rotated a. Therefore, since c rotates half as much as a does and half as much as b does, the total rotation of a and b together is twice that of c. In the pulleys (Fig. 4) note that if one of the pulleys, say, a, is held stationary and b is pulled down, c would move up twice the distance that b moved down. As a result the total of a and b is one half of c.

In the above examples we used a and b as the independent variables. By using a and c, or b and c, as our independent variables, we can easily change the process to subtraction. Also, arrangement for subtraction can be made by merely reversing the direction of movement of a or b.

In the illustrations we did not show how the variable quantities were indicated. For linear movement we can attach a pointer which moves across a calibrated scale. For rotary motion we can attach a pointer to the shaft and have the pointer move around a calibrated face—like the hands on a clock. There are many other means of indicating an amount of mechanical movement on a calibrated scale. We are not interested so much in knowing all the different ways as we are in realizing that such a procedure is possible.

Figure 5



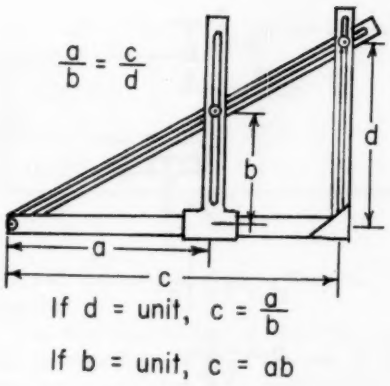
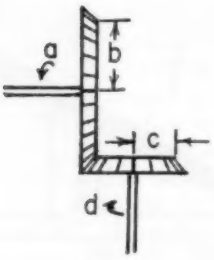


Figure 6

Just as in the case of addition and subtraction, there are many mechanical means for doing multiplication and division. Only a few simple illustrations are given. The accompanying figures show four of them.

The sliding scales (Fig. 5) are the same as the C and D scales on a slide rule and, therefore, should need no explanation. A linkage (Fig. 6) works on the principle that a line parallel to one side of a triangle divides the other two sides proportionally. A familiar linkage multiplier is the pantograph, which is used to draw pictures proportionally larger or smaller than the original. In the gearing mechanism (Fig. 7), and likewise in the pulley mechanism (Fig. 8), the amounts of rotation of the shafts are inversely proportional to their wheel diameters. This, therefore, can be the same as multiplying or dividing the amount of rotation by some constant.

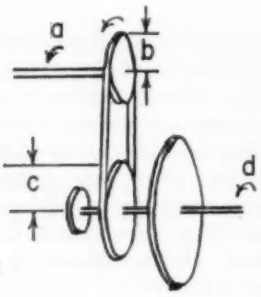
Logarithmic values can be generated from the lateral motion developed by rotating a logarithmic spiral cam. Trigonometric values, such as the sine and cosine functions, can be made from the lateral moving end of a shaft connected



$$d = \left(\frac{b}{c}\right) a$$

$$a = \frac{d}{\left(\frac{b}{c}\right)}$$

Figure 7



$$d = \left(\frac{b}{c}\right) a$$

$$a = \frac{d}{\left(\frac{b}{c}\right)}$$

Figure 8

to the circumference of a wheel; the connecting rod of a steam locomotive illustrates this type of motion. Mechanical integration and differentiation, which we will not try to explain here, are possible with a device known as the ball and disc integrator.

Electrical circuits are also used extensively in analog computers. Any discussion of these circuits would involve considerable explanation of what resistors, coils, condensers, and tubes do in the circuits. We will therefore give a few common examples which may help you to realize the possibilities. An ordinary flashlight with two or three batteries is a simple illustration of electrical addition. In putting them in series we add their voltages, so that the voltage applied to the light is the sum of the individual voltages. Also, you have undoubtedly watched an ammeter in a car add the currents used as you turn on the car radio, then the car heater, and, finally, the headlights. When you turn one of these off, the ammeter automatically subtracts some of the current and shows the remainder.

The speed control of a toy electric train is an example of electrical multiplication and division. Although the speed control (a variable autotransformer) uses 110 volts, the amount of voltage used for the train is varied from zero to about six volts. We can say, therefore, that we have divided the applied voltage by some amount or that we have multiplied the applied voltage by some fractional amount. The volume control of the radio is another device that is actually doing multiplication or division. Electrical circuits have also been devised which will work out analog problems in logarithms, trigonometric functions, differentiation, and integration. In each of these electrical computations we are using continuous values, determining in each case an answer of "how much."

Digital computations

Now let's investigate the digital computers and see how we feed "how many" into the computer and how the machine processes these numerical quantities. Although many of the modern digital computers are mechanical, this discussion will be mainly concerned with the high-speed electronic digital computers.

The mechanical digital operations are similar to the mechanical analog operations except that arrangments are made so that each equipment motion is in whole-number multiples of some unit of motion. A popular way of doing this is by using ratchets. A ratchet is a spring-loaded catch which stops a gear or wheel when the rotating energy is stopped, and the catch fits into a groove on the gear or

wheel. For instance, the winding mechanism on a clock has a ratchet which will stop the gear at the nearest groove back from where you stopped winding. Also, we can mechanically move along a straight line in discrete steps. A fine illustration of this is the type of car jack where each time that you pump the handle the jack moves up one notch. An adding machine makes use of stops; when we push a key on the machine a bar moves out and stops a rotating cylinder after it has completed the indicated number of unit motions.

It should be noted that when we add numbers with a digital computer it is basically a process of counting. We start with a given number and count up the number of digits in the second number. In subtraction we count down from the first number. Multiplication is repeated addition and division is repeated subtraction. Likewise we find that we can produce logarithms and trigonometric functions by the use of series, which again are merely continued additions and subtractions.

We normally do not think of a desk calculator as being slow; in fact it can divide one ten-figure number into another and come up with the answer in four or five seconds, a process which would take many of us at least that many minutes. The few seconds, however, seem comparatively slow when we realize that the relays used in electronic digital computers act in about a hundredth of a second. Then again, the relays appear slow when we are told that the vacuum tubes used in digital computers record an electrical change in less than a millionth of a second.

The trouble with all this wonderfully quick action is that the relays and vacuum tubes have only two possibilities: the relay is either open or closed, and the vacuum tube is either conducting or not conducting. It would be simpler if these devices could have ten possibilities so that we could use them directly with our decimal notation, but since they do not, we have to have some means of converting our

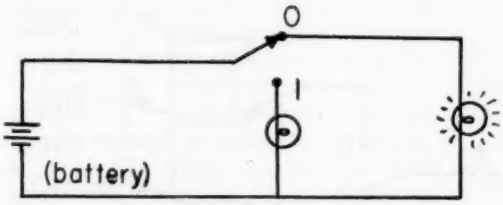


Figure 9

decimal values into values that can be applied.

The two possibilities of the relay and the vacuum tube suggest the use of the base two—and that, to a large extent, is what is used. We are familiar with the type of binary system in which 0 and 1 are the digits and each combination of 0 and 1 in the binary system represents some number in our decimal system.

To give an idea of how we can use the binary system with relays or tubes, it must be understood that we can connect a relay so that when it is closed (energized) it lets current flow along a certain circuit, and when it is open (de-energized) it lets current flow along a different circuit. Similarly, two vacuum tubes can be connected so that when one is conducting the other is not conducting, and vice versa. We can represent this relay action or this tube action schematically by a single-pole double-throw switch, the kind of a switch that we use to switch current from one circuit to another. Figure 9 shows such a switch connected from a battery to one of two lights. The way the circuit is now arranged, the light that is on could be lighting up a 0 (zero) on a panel, and when the switch is thrown the other way, the other light could light up a 1 on the panel. In place of the lights we could have a device in each branch which would punch a card in certain spots which represent 0 and 1 respectively. The point is that one position of the switch represents 0 and the other 1.

We can also combine several switches in combinations of 0's and 1's, as shown in Figure 10. The dotted lines in the

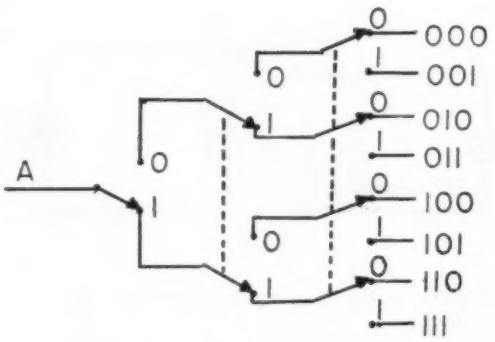


Figure 10

diagram indicate which switches are mechanically ganged together so that they, as a unit, all move the same way at the same time. The up-position of the switches represents 0's and the down-position, 1's. The switches are shown with the first two down making a path from *A* to 110. This could be the binary number 110 or it could represent the decimal number 6. Each additional column switch that we add to the diagram doubles the number of permutations. You can readily see, therefore, that by using sufficient columns we can set up some extremely large numbers with which to work.

Although using the binary system involves many more computations than the decimal system, that does not bother us when we realize how much faster the vacuum tubes work than do ordinary adding machines. Another advantage of the binary system is that its basic idea fits in so nicely with everyday occurrences. Think of how many daily situations you come across which have two possibilities; many of them will boil down to a no or yes, up or down, black or white, backwards or forwards, is or is not, positive or negative, more or less.

Not only do we have the "either/or" possibilities, but we also have situations which can best be expressed by the term, often used in technical correspondence, "and/or." This expression refers to anything that applies to both of two possibilities along with anything that applies to either of the two possibilities. Another combination is simply all those things which apply to both of two possibilities. These two combinations that we have suggested in this paragraph are actually used in some digital computers; the first is referred to as an "or" combination and the second as an "and" combination. Figure 11 shows two electrical circuits in which these can be performed.

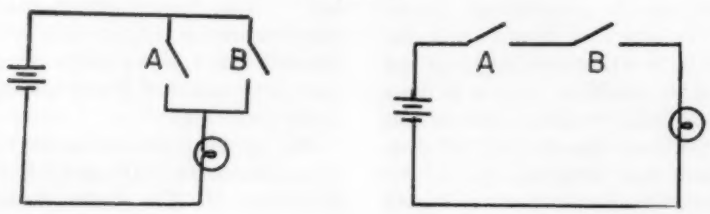
Note that in the "or" circuit the light goes on when switch *A* is closed, when switch *B* is closed, or when both switches are closed. In the "and" circuit, notice that both switch *A* and switch *B* have to be closed in order for the light to light.

We can use the analogy of letting "0" stand for a switch that is open or a light that is off and letting "1" stand for a switch that is closed or a light that is on. If we also let "+" stand for the "or" combination and "×" for the "and" combination, we arrive at the following relationships:

$$0+0=0 \qquad 0\times 0=0$$
$$1+0=1 \qquad 1\times 0=0$$
$$1+1=1 \qquad 1\times 1=1.$$

These are the basic relationships of Boolean algebra. These equalities are the

Figure 11



starting point for working many more involved circuits and problems in digital computers.

We are interested in the "or" and "and" combinations because many of the situations with which we come in contact and which we would like to study can be put into one of these forms. A simple illustration of an "or" circuit could be found in a large corporation that had punched-card records on all its employees when the corporation wanted to pick an individual for a special assignment who had either a degree in a certain specialized field, or at least three years experience in a certain type of work, or both. An "and" circuit could be used in the same corporation to pick out a person who had had experience as both a welder and an electrician. You can undoubtedly realize that by placing many other restrictions on the type of persons that we want to pick out from the group, we can have more complex circuits which are combinations of the "or" and "and" circuits. In all of these cases we are working with individuals and coming up with answers as to "how many."

The above illustration of a use of a digital computer is very basic. These computers can be used in practically any situation involving numerical quantities and can perform an unbelievably large number of mathematical operations with these quantities. The big problem is to determine how to put the information in such a form that it can be fed into the computer, how to build or set up the computer so that it will handle the information, and how to extract the answer. The process of "programing data" for computers is becoming a bottleneck in computer use.

There are many computers already built, and many more being built. There are many large corporations with problems solvable by computers. What is needed are the people who can set these problems in a mathematical form that can be introduced into the computer. These people are primarily mathematicians—and the number needed, or going to be needed, is astounding. When one considers that about fifteen years ago there were only a few hundred people working on computers, that now there are over a hundred thousand, and that in five to ten years the figure will be over a million, one wonders where so many people with the necessary background will come from.

Although many companies train their own employees to a certain extent, the individuals so trained must have a special mathematical background before the companies even consider them for such training. These companies are going to require more and more of their applicants for these specialized jobs. If we are to produce such computer-wizards, it is going to be up to us as mathematics teachers to teach as *many* as possible as *much* as possible about the modern mathematics they are supposed to know.

Bibliography

Alliston, W. "A Broad Look at Analog Computers," *Control Engineering*, II, No. 2 (February, 1955), 53–57.

Davidson, G. M. "Basic Math with AC Analogs," *Control Engineering*, II, No. 3 (March, 1955), 57–66.

Korn, G. A., and Korn, T. M. *Electronic Analog Computers (D-C Analog Computers)*. 2d ed. New York: McGraw-Hill Book Company, Inc., 1956.

Richards, R. K. *Arithmetic Operations in Digital Computers*. New York: D. Van Nostrand Company, Inc., 1955.

Soroka, W. W. *Analog Methods in Computation and Simulation*. New York: McGraw-Hill Book Company, Inc., 1954.

Stibitz, G. R., and Larrivee, J. A. *Mathematics and Computers*. New York: McGraw-Hill Book Company, Inc., 1957.

Svoboda, A. *Computing Mechanisms and Linkages*. New York: McGraw-Hill Book Company, Inc., 1948.

A high school digital computer program

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The lead article in this issue has emphasized the extensive use of computers. This article tells what one teacher has done with a homemade digital computer for the high school.

THE CONTENT of high school mathematics over the last several decades has shown changes which reflect the changing needs educators have recognized. This last decade has seen the growth, a phenomenally accelerated growth, of a new tool, the automatic digital computer. I feel that the large-scale application of digital computers to the problems of business and science during the last few years is now another important factor which educators must recognize in planning the high school mathematics curriculum. The purpose of this paper is to help readers who see the need for instruction in the fundamentals of automatic computation plan a suitable high school course. The course to be described in this paper has now been taught for four successive terms to selected college-preparatory seniors at the Hutchinson Central Technical High School.

The major objectives of this one-term Digital Computer Mathematics course are: (1) to tap the unused mathematical potential of some of our more gifted pupils; (2) to acquaint a number of our seniors with fundamental concepts of automation, and particularly with the application of these concepts to automatic digital computers; and (3) to generate interest in mathematics and science in our student body.

The course of study set up for our computer class consists of theory and laboratory work. After six weeks of studying theory five periods per week in order to establish a background, the class gets two laboratory periods and three theory periods per week until the end of the semester.

The theory part of this course attempts to give the pupils a broad basic understanding of the fundamentals of computing. We begin with a study of our number system to base ten, the idea of place value of digits, and the polynomial form of numbers. We then branch out into number systems in general, covering base, number of symbols needed, and the conversion of numbers from one base to another. The class then concentrates on the binary number system and its relation to two state devices. After some further work in conversion, we drill addition, subtraction, multiplication, and division in this system to free our pupils' minds from the rigid mold of base ten.

The second major unit on theory concerns the history, the uses, and the construction of computers, and in this unit, we touch upon the logic, the circuits, the speed, and other characteristics of early and, then, more modern computers. This is followed by a study of the units of a modern automatic computer. At this point simple wiring diagrams are introduced and studied. The pupils learn to devise various signal-routing networks and gates, and then set up networks which perform one column and then multicolumn addition, subtraction, and multiplication. I find that high school pupils display considerable insight into the logic of these circuits. In this course our pupils

are limited to making circuits employing relays, although electronic devices, flip-flops, etc., are discussed. It is interesting to note that elements of Boolean algebra integrate very well with the work of this unit. The members of the class study the basic postulates of circuit algebra and do some work in simplifying switching functions by the use of both the postulates and the construction of truth tables.

Because the next unit studied, programing for an automatic computer, is such an important part of this course, it will be described in detail in a separate section at a further point in this paper.

The last major unit of the theory work of the computer course consists of several topics from the field of numerical analysis. The reader will find much of the material in this field to be on the college level, so that, to some extent, this material has to be predigested and simplified before being offered to pupils in high school. The pupils in this class begin this unit with a study of approximate calculations, accuracy, precision, and the absolute, relative, and percentage errors in approximate calculations. The class then continues with a study of differences of various orders, the relation between the degree of the function and the order of the differences which become constant, uses of differences, simple interpolation, interpolation by divided differences, and the use of Newton's formulas. This study is followed by some elementary work in numerical integration as applied to the area under a curve using the trapezoidal rule, Simpson's rule, and Weddle's rule. We also study the derivative of a function and the determination of the real roots of an equation by Newton's method. And finally, some work is done with the method of least squares and curve fitting of data. I find that this work falls within the interest and ability of the pupils, especially when it can be indicated how this material ties in with automatic computation.

This part of the course can be taught by having the pupils program for an imaginary computer. A very worthwhile and interesting set of lessons can be developed after assigning to this imaginary computer various possible operations and codes. However, I feel that it is very desirable that the programing in such a high school course be concrete, and that the pupils in the course be able to run their programs on an actual specific machine, as well as learn general principles of programing. Such a machine can be built by high school pupils. The Hutchinson Central Technical High School digital computer was built by the pupils, has been in operation two years, and is used by the members of this class to execute their programs.

The unit on programing for an automatic computer, and in particular, programing for our machine, forms a large part of the theory work, and is also the core of practically all the laboratory work of this course. In the laboratory work in our course, the young programer can construct his program from the following "building blocks" or orders which our automatic computer can execute: (1) read in data into any register, (2) transfer information to or from any register, (3) clear any register, (4) the arithmetic operations —add, subtract, multiply, and shift right one place, and (5) the logical operations— test whether the number in the output register is positive, negative, or zero; test whether there is overflow in the output register; test whether certain toggles are in the plus or minus position; and lock or open certain gates and thus choose and follow one of several alternate subroutines on the tape, depending upon any of the previous determinations.

We begin the unit on programing for the automatic digital computer with a study of the orders our machine can execute and the representation of these orders as binary numbers to determine the holes in the tapes used in the automatic operation of this computer. Then the pupils analyze easy problems and break down their solutions into the simple orders

the computer can follow. The manual input of the computer is very useful at this point in demonstrating the execution of the orders in the programs the pupils obtain. Now, using the technique of round-table discussion, the pupils of the class analyze the more advanced problems given them, and build up the programs for their solution, the tapes of which are then run on the computer, keeping the work concrete. Next, algebraic transformations and various sequences of orders that minimize the number of orders in a program are studied by the class and applied to their problems. Pitfalls and "bugs" are discovered, discussed, and listed for future avoidance. When enough experience has been gained by each pupil from the common discussions, he is ready to embark on the analysis of his individual original problems and the programing of their solutions. For maximum credit, each problem set up by a pupil must contain branch points and cycling routines, must consist of a minimum number of orders, and must have no "bugs."

After a pupil has chosen a problem and had it approved, he uses a flow diagram to sketch the bold outline of his solution. Then, as he determines the successive orders of his program, he writes these orders into his program. During the several discussions of his program with his instructor, the pupil justifies his program, corrects any errors he may find, and tries to shorten his program. When the pupil is satisfied with the program he has achieved, he uses the manual input of the computer to test it. Then the pupil codes his program and again tests it by setting the successive code numbers manually on a device we built which inserts individual code numbers into the tape reader. In this procedure the pupil is able to check all his orders, one by one, including those which involve the logical operations. If he finds no errors, he is permitted to cut his program on the special hard-surfaced stock which we use for our tapes. The more interesting tapes are reproduced by our pupils on plastic sheets for permanence.

As part of this unit, we also study briefly elements of programing and coding for card-programed machines reading from several alternate fields of instruction, and for internally programed machines providing for conditional and unconditional returns in their instruction cycles. The more gifted pupils are urged to do further work in programing and coding their solutions of simple problems for such commercial machines. The various operating manuals for these commercial machines are available, and our pupils get extra credit for this work. I hope that this rather detailed account of the programing we do has given the reader some idea of what can be done with this phase of the work in a digital computer mathematics class.

A brief description of the training computer built by our pupils may be of interest to the readers of this paper. The Hutchinson Central Technical High School digital computer is an electromagnetic relay machine operating in the binary number system. Its input used for automatic operation is a separate ten-channel magnified tape reader, which plugs into the main machine by means of a twelve-contact plug. The storage section, the arithmetic section, the control section, and the output make up the main machine.

The various units of these sections of the main machine are mounted on three tempered Masonite panels, each 48 inches long by 40 inches wide by $\frac{1}{8}$ inch thick. Each panel is sandwiched between the front and back sections of its wooden frame, thus making a very rigid structure and also providing a recessed space 2 inches deep on both faces of each panel for mounting components. The three framed panels are joined by two 40-inch piano hinges so that the machine may be opened like a book to study its circuits. Four hasps keep the machine locked and add to its rigidity. The frame is mounted on gliders to make it movable. The components of the machine include 273 relays, 62 panel lights, and 45 switches.

The author of this paper designed the plans and circuits for this training digital computer, and the Buffalo Board of Education, upon recommendation of the administrative staff, purchased the necessary materials. The pupils in the Tech Math Club at our school constructed the frame, mounted the components, and wired the circuits of our digital computer.

Readers interested in computer mathematics may acquire a more complete understanding of the procedures used in programing for an automatic digital computer by making a careful study of the following two programs. They illustrate the very simple orders which must be given the Hutchinson Central Technical computer. The pupils in our computer mathematics class discuss and construct these programs as part of their group work while gaining experience in programing and coding for our machine.

The very first program we set up as a group is the simple set of orders which directs our machine to find the average of two positive integers, a and b. After very little discussion, the class determines the following procedure: (1) make a tape containing the eight orders listed below, (2) use the keyboard to enter integer a in register one and integer b in register two, and (3) pass the tape through the tape reader. The following eight orders on this tape cause the machine to calculate the average and place it in output register C:

1. transfer the contents of register one to operator register A
2. transfer the contents of register two to operator register B
3. add (Note that the answer appears in output register C.)
4. stop the machine if addition overflows
5. clear operator register B
6. transfer the contents of output register C to operator register B
7. clear output register C
8. shift right one place (Again, the answer is in output register C.)

After the pupils gain experience in programing, they return to this program and replace the stop order in this program by a branching routine which takes care of overflow.

The second program is a more advanced routine our pupils set up for determining a positive real root of the equation $x^2+bx-c=0$. The solution of this problem includes as a special case, finding the square root of a number. To keep the programing simple, only roots equal to or less than seven will be determined by this program.

To prepare the pupils for this problem, I first introduce some discussion of converging sequences, and in particular, the sequence of left end-points of intervals formed by successively bisecting an interval in which a required root lies. We then sketch the quadratic function, $y=x^2+bx-c$, using several values of the constants, b and c, in order to see the behavior of this function, where it is positive, and where it is negative.

Then a flow diagram is made and after a number of trials and corrections, the pupils construct a program similar to the following: (1) use the keyboard to enter integer b in register one and integer c in register two, (2) enter the value *four* in register three, and (3) pass a tape containing the following thirty-five orders through the tape reader, cycling this tape a total of three times. Note that if the required root is an integer, it will appear in register *six* at the end of the third cycle, but if the required root is not an integer, only the integral part of it will appear in register six. The orders on the tape are as follows:

1. transfer the contents of register three to operator register A
2. transfer the contents of register six to operator register B
3. merge the contents of operator registers A and B (This is *logical* addition.)
4. clear operator register B

We note that orders 1 to 4 set up a new trial value of x for the second and third cycle but make no change at the first cycle.

5. transfer the contents of register one to operator register *B*
6. add
7. stop the machine if the addition overflows
8. clear operator register *B*
9. transfer the contents of output register *C* to operator register *B*
10. clear output register *C*
11. multiply
12. stop the machine if the multiplication overflows
13. clear operator register *A*
14. clear operator register *B*
15. transfer the contents of output register *C* to operator register *B*
16. clear output register *C*
17. transfer the contents of register two to operator register *A*
18. subtract

We note that orders 5 to 18 evaluate $y = -f(x)$.

19. clear operator register *B*
20. transfer the contents of register three to operator register *B*
21. clear register three
22. lock control gate *X* if the number in output register *C* is *negative*

We note that order 22 is a branching order of this program.

23. clear output register *C*
24. divide the contents of operator register *B* by *two*
25. merge the contents of operator register *B* and register six

We note that orders 23 to 25 make up the first alternate subroutine.

26. lock control gate *Y* if control gate *X* is open
27. open control gate *X* if it is locked
28. clear output register *C*
29. divide the contents of operator register *B* by *two*
30. open control gate *Y* if it is locked

We note that orders 26 to 30 make up the second alternate subroutine and that we now return to the main program with order 31.

31. clear operator register *A*
32. clear operator register *B*
33. stop the machine if the number in output register *C* is *zero*
34. transfer the contents of output register *C* to register three
35. clear output register *C*

In the execution of the above program, it can be seen that any single cycle of the tape causes a right shift of one place in the contents of register three, but may or may not alter the contents of register six, depending upon the contents of register *C* when order 22 is executed. The successive values of the contents of register *six* form a sequence converging upon the real positive root of the given equation.

Field trips and films

I find a program of visits to industrial establishments using automatic computers to be an important part of our course. We plan one such visit each month, each visit lasting about an hour and a half. We find these visits interesting and informative, and have always found personnel and management most co-operative. Readers may be interested to learn that such national corporations as International Business Machines, Remington Rand, Bell Aircraft, National Gypsum, etc. have welcomed visits to their Buffalo offices, as have many local businesses such as Cornell Aeronautical Laboratory.

The reader will also find that films can do much to furnish background for the course. Splendid films concerning automatic computers have been produced by International Business Machines, Remington Rand, Moore Business Forms, etc. and can be rented at no cost or for a very nominal fee.

Conclusion

I believe this course to be a step toward the fuller utilization of the scientific and the mathematical abilities of young people. It is possible for other schools to adopt the ideas I have outlined and so give their pupils some very rewarding experiences in an important and interesting branch of mathematics, machine computation.

A modified curriculum for capable students

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The results of an experiment with first-year algebra in grade eight are presented for your consideration.

ALL OF US, as teachers of mathematics, have given serious consideration to the problem of accommodating in our classes the students who could be classified as rapid learners. And, like you, the teachers of our mathematics department have tried to devise methods of instruction which would be flexible enough to permit the rapid learner to progress as rapidly as his ability in mathematics will warrant. It is the purpose of this paper to describe some of these methods. Before I begin, I want to make it perfectly clear that the methods to be described are not to be interpreted as constituting a complete solution, but are merely the efforts of one school directed toward this problem.

First, I will explain our efforts to provide for the rapid learners in the eighth grade at the Westside Community School. At the beginning of the school year 1955–56, twenty-five students were selected to take first-year algebra. This selection was based on intelligence, achievement in arithmetic, and desire to take this algebra course.

As the semester progressed, it became obvious, even though the class was a select group, that there were great differences in ability. In an effort to assure that each student would have the opportunity to progress at a rate determined only by his ability and desire, the class was divided into three groups. The purpose of this action was discussed with the students. It was explained that the first group would work independently of the class, that is, each student would set his own pace. The only requirement was that the members of this group must always be in advance of the second group. When a student had completed a unit, he would be given an examination to determine his grade. The second, and largest, group was to continue receiving instruction in the same manner as before the three groups were formed. The third, and smallest, group was to progress at a slower rate than the second group, and to receive as much special help as possible. After this explanation, the students were given the opportunity to select the group of which, in their opinion, they should be a member. There seemed to be no hesitancy on the part of the students who should have selected the third group to do so. It was necessary to call in only one student for a conference concerning his choice. He had selected the group which was to continue in the same manner. In the opinion of the teacher, he should have selected the group which was to work independently. As a result of this conference, he agreed to move to the recommended group temporarily. He soon became a permanent member of this first group.

To give the individual help required, three fourth-year mathematics students were selected as student aids. The selection was based on intelligence, achievement in mathematics, and desire to participate. It was the responsibility of these

student aids to attend the class each day. They were seated near the back of the room where each had a table that would accommodate two people. During the time that the teacher was talking to one group, the members of the other groups were free to go to one of the student aids for help. Before giving this help, each of the student aids was instructed to ask for some evidence that the student had made a sincere effort to solve his own problem. The student aids were also asked not to do the work for the student, but to help him solve the problem for himself.

Since this algebra class met the last period of the day, it was convenient for the teacher and the student aids to hold discussions after school. They discussed the best ways to question a student requesting help to enable him to solve his problem. They also talked of ways to stimulate interest and participation, progress of the students, and other matters of importance to the class. The information contributed by the student aids helped the teacher know and understand the members of the class better than would have been possible without their contribution.

To evaluate the achievement of this class, we gave the same final examination that was given to a class of twenty-five comparable ninth-grade students. After the test was scored, it was determined that fifteen of the twenty-five highest scores were earned by eighth-grade students. Since the teacher of this ninth-grade group was recognized as a superior teacher, it was concluded that the students of the eighth grade were motivated to achieve more nearly up to their ability, because they were members of a select group. They were challenged to produce.

Although credit for high school mathematics was given for this course, the members of this class, as freshmen, registered for second-year algebra for the 1956–57 school year. No effort was directed toward keeping this group together as a unit. The members of the group were placed in a section of second-year algebra on the basis of their achievement the previous year. In our school, the students who intend to take three years of high school mathematics are urged to take two years of algebra before taking geometry. With this arrangement it is possible for a ninth-grade student to be in a section of second-year algebra in which there are sophomores, juniors, and seniors. The ninth-grade students are achieving as well as the students of the upper grades of comparable ability in their respective sections.

Now, about the future of this group. It is tentatively planned that during their sophomore year these students will take geometry. We believe that the purpose of teaching Euclidean geometry is to give the student an appreciation of the structure of a logical system. We also believe that it is not necessary to prove eighty or ninety theorems to demonstrate this structure. Therefore, we are going to attempt to accomplish this purpose in approximately one semester. During the second semester of the sophomore year we plan to introduce trigonometry and coordinate geometry, and in the junior year we plan to offer a course that will include such topics as trigonometric analysis, linear and quadratic functions, sequence and limit, and differentiation.

The senior-year program will be strictly on an individual basis. Each student, with the help of a mathematics teacher, will select a project or course of study that is related to his personal goal. The completion of this activity will be the responsibility of the individual student. The faculty adviser will encourage and give direction to the student in regularly scheduled conferences.

Perhaps this sounds like a very ambitious program. But these plans were not made without some evidence that it would be possible to achieve our goal. During the school years 1954–55 and 1955–56, eleventh- and twelfth-grade mathematics students whose scholastic records indicated they were rapid learners were allowed to work at their own rate, independently of

the class. The students who elected to follow this plan were able to complete work of sufficient quantity and quality to warrant at least one semester's more credit than the members of the group who did not accept this opportunity. The success of these students on competitive examinations and their placement in freshman mathematics classes at the institutions of their choice provide a part of this evidence. In addition, the parents of the community were enthusiastic in their acceptance of this method of instruction. We considered this evidence sufficient to extend this plan to the junior high school.

To solve such problems as teacher load, scheduling, and the granting of credit will most certainly be a challenge. But in spite of these and other problems, we feel that the first step has been taken in providing for the differences in ability of our students.

What's new?

BOOKS

SECONDARY

Business Mathematics (4th edition), Cleon C. Richtmeyer and Judson W. Foust. New York, McGraw-Hill Book Company, Inc., 1958. Cloth, xii+412 pp., $5.75.

The Teaching of Mathematics. Incorporated Association of Assistant Masters in Secondary Schools. New York, Cambridge University Press, 1957. Cloth, ix+231 pp., $3.00.

COLLEGE

Analytical Conics, Barry Spain. New York, Pergamon Press, 1957. Cloth, ix+145 pp., $5.00.

Integral Equations, S. G. Mikhlin (translated from the Russian by A. H. Armstrong). New York, Pergamon Press, 1957. Cloth, xii+338 pp., $12.50.

MISCELLANEOUS

An Adventure in Geometry, Anthony Ravielli. New York, The Viking Press, 1957. Cloth, 117 pp., $3.00.

Soviet Education for Science and Technology Alexander G. Korol. Cambridge, Technology Press of Massachusetts Institute of Technology; and New York, John Wiley and Sons, Inc., 1957. Cloth, xxv+513 pp., $8.50.

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Teaching the etymology of mathematical terms[1]

T. F. MULCRONE, S.J., *Loyola University, New Orleans, Louisiana.*

Hints on how to make use of word origins and meanings in a mathematics class.

MOST GOOD TEACHERS OF MATHEMATICS, like good teachers of any subject, do make use of the etymology, that is, of the literal, original, or root meaning of words, in the course of their teaching. But I do not think that we mathematics teachers are using to best advantage this technique that would certainly improve our teaching. Explaining the root meanings of the words we mathematicians use makes for a better understanding of the mathematics we teach. Such explanations also contribute to that broadening, humanizing benefit which more properly but certainly not exclusively is the responsibility of language and history teachers—a benefit which seems to be regrettably deficient in the formation of many students at the present time.

I am convinced that the stimulation of a student's interest in language is the responsibility and the gloriously rewarding opportunity of every teacher. In a quite personal way the terminology used by the mathematician reflects the wellsprings of history in which his concepts and creations have their motivation and inspiration. The little time and effort it takes to awaken and cultivate in ourselves an eager awareness of the colorful meaning of words—meanings which frequently tell very revealing stories behind concepts or whole areas of mathematical thought—will reward us, the teachers, with motivation and inspiration. And once we ourselves have the habit of delving into the root meaning of words, our students also will run with a measure of relish to those bracketed hieroglyphics which precede dictionary definitions and which are, in the matter of mathematical terminology, of far greater value than dictionary definitions themselves. If we can persuade our students to be dictionary-minded in the sense that they are eager to get to the root meaning of the words they meet, then the mathematics we teach will be much better assimilated and much more culturally integrated with their other studies and their lives as a whole.

It should be noted that for the teacher to cultivate this awareness of the root meaning of words, it is not necessary that he have any acquaintance or any great interest in foreign languages. None of the books which deal with word lore makes such a demand on the reader. Indeed, the fact on which the value and infallible success of this teaching technique rests is the remarkable number of foreign roots, especially Latin (L.) and Greek (G.), that all of us, and most of our students, already know. The fact is that Latin, directly or through the Romance languages, particularly French, comprises fully one half of our English vocabulary. And Greek furnishes over one tenth of our vocabulary. These figures would have to be raised considerably were we to restrict ourselves to

[1] Delivered on February 15, 1957, in Biloxi, Mississippi, at the joint meeting of the Louisiana-Mississippi section of the Mathematical Association of America and the Louisiana-Mississippi branch of The National Council of Teachers of Mathematics.

those terms which are properly mathematical.

What I would like to do is to suggest some openings or leads whereby a short reference by the mathematics teacher to the etymology of mathematical terms that occur in our courses will serve as an aid in one or more of the following:

a) a better understanding of the mathematics taught,
b) the remembering of definitions,
c) the identification of concepts in their historical setting, and
d) the generalization and unification of mathematical types.

It goes without saying—but students must be reminded of this—that we do not get precise mathematical definitions by consulting an English dictionary. The meaning of mathematical terms can be had only from mathematicians themselves. This accounts largely for the fact that of all scientific language that of the mathematician is the most precise. But at that it is not static[2] or perfect. It will never be such as long as mathematics is alive! Nevertheless, a study of the etymology or root meaning of words does serve as an aid in one or more of the ways mentioned above.

One of the most encouraging and helpful revelations a student can be given is the assurance that he really knows already the roots that go together to make up a word which appears, at first sight, to be a strange word. The mathematics teacher can do this by pointing out that the words or the roots of the words which the mathematician uses appear also in the writing and conversation of all who employ the English language. Thus *cardioid* [G. *kardioeide*—heart-shaped: *kardia*—heart; *eidos*—shape] suggests *cardi*ac, *cardio*gram, *cardio*graph, *cardio*logy, *cardi*tis, . . . and cellu*loid*, spher*oid*, ellips*oid*, anthrop*oid*, mast*oid*, . . .

Once one meets any of the many words that scientists have adopted from classical languages (has not the language of the electrical engineer been called "Schenectady Greek"?[3]), there is no trouble in finding other words with one or more of the same roots. With less mature students the teacher does well to coax from the class derivatives from among non-mathematical words. Thus *geometry* [G., L. *geometria*—land measure: G. *ge*—earth; *metron*—measure] suggests *geo*graphy, *geo*logy, . . . and (Hint! There are some on the instrument board of an automobile!) speedo*meter*, am*meter*, *meter*, volt*meter*, centi*meter*, . . . With more advanced students I have found the following approach a very useful entree for short references to more advanced mathematics and more generalized or abstract concepts. The word *unit* [L. *unus*—one; *unitas*—unity], for example, suggests *uni*ty, *uni*que, *uni*form, *uni*lateral, . . . When the class can name no more derivatives the teacher can suggest, say, *uni*cursal, *uni*modal, *uni*verse and *uni*on. Such words will be drawn from the teacher's reading about (not necessarily mastery of) more advanced mathematics. In this lies a welcome opportunity. The mention of *universe* and *union* is a splendid occasion to say a little about the concept of a set, or the generalization of addition, or topology.

Having mentioned the term "unit," it may be noted that those mathematical terms which involve Greek or Latin numerals provide the teacher with very many opportunities to give the class an introduction to or a link with other branches of mathematics or some historical incidents,

[2] One can observe the crystallization of terminology at work in modern mathematics by noting how the interested reader frequently has to contend with lack of agreement among authors on terminology until eventually a manner of expression that is considered appropriate prevails. It is significant that modern mathematics, while still introducing words derived immediately from Latin and Greek, is also taking over many short, common English nouns and adjectives, some of them Anglo-Saxon, to represent new mathematical concepts. In the study of topology, for example, one meets such terms as *block*, *clan*, *floor*, *knot*, *mob*, *nice*, *roof*, *spread*, *tame*, *tower*, and *tree*. See Edward Kasner, "New Names in Mathematics," *Scripta Mathematica*, V (1938), 5–14.

[3] Mario Pei, *The Story of English* (Philadelphia: J. B. Lippincott Co., 1952), p. 107.

while at the same time permitting him to highlight the mathematical term itself. The term *triangle* [L. *triangulum: tria*—three; *angulus*—angle] suggests *trigonometry* [G. *trigonon*—triangle: *tri*—three; *gonia*—corner, angle; *metron*—measure]; the *quaternion* derives from the Latin *quaterni*—four each, by fours; *quattuor*—four. Again, the Latin numerals *septem*—seven, *octo*—eight, *novem*—nine, and *decem*—ten, appear as roots in several of our mathematical terms. How easy it is to teach the meaning of these roots if we point out that our months *Septem*ber, *Octo*ber, *Novem*ber, and *Decem*ber were respectively the seventh, eight, ninth, and tenth months of the Roman year.

Another easy yet very educational type of correlation is the noting of instances where two or more of our mathematical terms come from a common-idea origin. Thus *foot, palm, span, digit* (L. *digitus*—finger, toe, inch) are all parts or equivalently parts of the human body which serve as measures. To this group belongs the term *mile*, which is an abbreviation for the Latin *mille passuum*—a thousand paces. Again, we have *rod, pole, perch*, and *yard*, which were originally simply measuring sticks.

Much of our terminology, especially in geometry, stems from the shape of a curve or the manner in which the curve is drawn. The *catenary* [L. *catena*—chain], for example, is the locus of the hanging chain. And the *tractrix* [L. *tractum*—drawn, dragged] may be described as the path of a particle P that is dragged along a plane by a string QP of fixed length as Q moves along a fixed line not through P. Closely related to this locus is the *courbe du chien*, the "curve (path) of a dog" chasing a rabbit![4] And nothing will capture the interest of a calculus class more readily than to announce that you are going to consider in class the *osculating* circle [L. *osculatio*—a kissing] of a given curve at a designated point on the curve. Such a circle—if it is to be worthy of its name—obviously has a higher degree of contact with the given curve at the designated point than any other circle, and hence this circle must have its first and second derivatives equal respectively to the first and second derivatives of the curve itself at that point.

The mathematics teacher should try to capitalize on the fact that the student finds it much easier to remember those notions which have in some way been associated with physical reality. The persons who gave the names *trochoid* [G. wheel-like], *cycloid* [G. circle-like], *epicycle* [G. like upon (outside) a circle], *brachistochrone* [G. in the shortest time], *tautochrone* [G. in the same or equal time] certainly wished to give meaningfully significant terminology that would be singularly graphic. To develop in some detail how the mathematics teacher might go about such associations, consider the term *normal* as used with reference to straight lines and curves in geometry. This is a term which is troublesome for many students. But if we recall that for the Romans the "norma" was a carpenter's square, and that even among the Romans there was the transferred meaning of a "rule" or "norm of action," then the terminology offers no further difficulty. I have asked students in what way they would proceed from a designated point to a given straight line and what they would define as being the distance from the point to that line. Invariably the discussion comes to this: that the perpendicular distance is to be defined as the required distance and that the line thus determined at right angle to the given line, which the Romans regarded as "normal," is indeed defined to be such for us as well.

The mention of the term *normal* suggests another way in which the student can be made aware of the inner, the root, significance of the terminology we employ. It is instructive and interesting to compare the ways in which we can say the same thing or closely related variations of

[4] Arthur Bernhart, "Curves of Pursuit," *Scripta Mathematica*, XX (1954), 125–41.

the same thing by using synonyms. Other ways of saying *normal* are *perpendicular* [L. *perpendiculum*—a plumb line: *per*—exactly, straight; *pendere*—to hang; *culum* is a diminutive—hence, a small object that hangs straight] and *orthogonal* [G. *orthos*—straight, but often in the sense of upright or at right angles; *gonia*—corner, angle]. Indeed, in mathematics we have many duplicate substantives drawn from both Greek and Latin: *arithmetic* [G. *arithmos*—number] and *numeration* [L. *numerus*—number; *numerare*—to count, number]; *heptagon* [G. seven-angled] and *septangle* [L. seven-angled]; *perimeter* (*periphery*) [G. *peri*—around; *pherein*—to carry] and *circumference* [L. *circum*—around; *ferre*—to carry]; *hypothesis* [G. groundwork, foundation, supposition: *hypo*—under; *tithenai*—to place] and *supposition* [L. *sub*—under; *ponere*—to put]. Another pair of terms having their first roots in common with "hypothesis" and "supposition" is *hypotenuse* [G. *hypo*—under; *teinein*—to stretch] and *subtend* [L. *sub*—under; *tendere*—to stretch]. For myself, I never could see the why and wherefore of the term *hypotenuse* until I came to think of it as a special case of "subtend," namely, when the subtending angle is a right angle.

Another rewarding practice is to correlate the meanings of words that are used both as mathematical and nonmathematical terms. Consider, for example, the word *matrix* [L. womb: *mater*—mother]. A short reference to the function of a mother's womb, and to what a printer's matrix and a dentist's matrix are, helps very much in putting over to a mathematics class the difference between a matrix and a determinant. Again, the *parabola* [G. *para*—alongside of (cf. L. *par*—equal); *ballein*—to throw], *ellipse* [G. *elleipsis*—a defect; *elleipein*—to fall short], and *hyperbola* [G. *hyperbole*—an excess: *hyper*—over, beyond] can be very significantly distinguished by noting that the etymology of these terms refers to the curves as sections of a cone, and also by correlating their mathematical with their rhetorical meaning. Finally, to cite but one more example, how much fuller a meaning does the concept of *interpolation* [L. *interpolare*—to furbish or patch up, to falsify: *inter*—between; *polire*—to polish] take on when we recall that the word meant for the Romans to polish up in between, to dress up, to embellish, and also to enlarge, corrupt, or alter a manuscript.

Frequently enough, the terminology we use affords much of an insight into the history of concepts, our words serving as a sort of cultural cement. The most obvious instances are those functions and loci that are named after noted mathematicians, for example, the *spiral of Archimedes*, *L'Hospital's rule*, *Hero's formula*, *folium of Descartes* [L. *folium*—leaf], and the *Gudermannian*. (The Latin *homobonus*—good man, incidentally, has set the pattern for a proper name in most languages. In French it is Bonhomme; in English we have Goodman.) Such examples as these afford the teacher an excellent opportunity for a few words about the person in question or the age in which he lived.

Less obvious instances in which the teacher can correlate mathematical terminology with some historical connotations or historical terms with some mathematical background are more difficult to detect, but they are extremely rewarding both to the teacher and the student. There are many such instances in mathematics. The term *statistics*, for example, reminds us that the branch of study originated as the science concerned with the assembling of data (economic, sociological) concerning the state [L. *status*—condition, circumstances, state]. Professor Jacques Hadamard has noted that "the first mathematical foundation in Greek science, geometry, was suggested by practical necessity, as can be seen by its very name, which means 'land-measuring.' "[5] The point he makes is true, although the origin of the word *geometry* is disputed. But such practical

[5] Jacques Hadamard, *The Psychology of Invention in the Mathematical Field* (Princeton: Princeton University Press, 1945), p. 125.

necessity is manifest even if, as is not unlikely, the Greek word was a translation of an Egyptian term meaning land measurement.

In the teaching of weights and measures, incidentally, we owe our students an explanation of the conventions that are followed today. Fortunately an explanation is often close at hand if we just go back to the root meanings of the terms we use. The troy and apothecaries' pounds, containing only twelve ounces, always raise a question in the students' minds. Actually, it is our sixteen-ounce avoirdupois pound that is a later development. Note how much light is thrown on the subject when we recall for our students that the English words *inch* and *ounce* both derive from the Latin word *uncia* (pronounce it "unshia," and you note the resemblance to the English derivatives) meaning one-twelfth part, especially a twelfth of a foot, a twelfth of a pound, a twelfth of a jugerum (a land measure), and a twelfth per cent a month (one per cent per annum).

Again, in introducing *pencils* of lines the mathematics teacher can point out that the Latin word *penicillum* denoted a small paint brush [L. *penna*—feather; *culum* is a diminutive—hence, a little feather, a quill], the forerunner of our pen and pencil, and that Sir Alexander Fleming aptly used the same word to designate the remarkable medicinal mold that appeared as a stem branching out into a brushlike end. And a by no means trivial example is the word *trivial* itself which is frequently used by mathematicians. Its etymology is disputed. Most etymologists regard the word as deriving immediately from the Latin *trivium*—a place where three [L. *trí*] ways [L. *viae*] meet, that is, found in the most frequented places, hence ordinary, common, on everybody's lips. Other etymologists take the term as referring to the *trivium*—the three preparatory disciplines, grammar, logic, and rhetoric, that were the paths to the *quadrivium*—the four finishing courses which in the Middle Ages were esteemed as constituting a liberal education. In either case we have in the term *trivial* a connotation that is singularly rich and significant in the picture which it gives us of the past. I always add, in mentioning this etymology in class, that as a matter of fact the *quadrivium* or finishing courses were arithmetic, geometry, astronomy, and music, and that the etymology of the word implies that these courses were not regarded as trivial in the past, and they should not be regarded as trivial now!

The terms *calculus, calculate, calculation* will have to be the last examples cited in this very rich field. The word *calculus* in Latin denotes a small stone, especially a pebble used by the Romans (and by many other peoples) as a counter to aid in computation. And in Late and Medieval Latin the term designated certain stone diseases. We speak in the same manner when we use expressions like gallstones and kidney stones.

To mention a historical term with a mathematical background, there is the use of the name "Dixie" as referring to the South. This name most probably (disputed origin) comes from a ten-dollar note issued by a New Orleans bank before the Civil War. On the reverse side of the note was a large French *dix*—ten, whence came "land of Dixies" or "Dixie land."

Finally, our students should be informed that, although mathematicians can rightly pride themselves on the general appropriateness of their terminology, there are some instances of concepts which have been questionably or inappropriately named.

The first instances that come to mind are some attributions such as the *Cardano solution* of the cubic and the *Simson line*,[6] and a fair number of honorary dedications due to the facetious Poincaré.[7] Even these, however, provide splendid opportunities

[6] R. A. Johnson, "Relating to the 'Simson Line' or 'Wallace Line,'" *The American Mathematical Monthly*, XXIII (February, 1916), 61–62.

[7] E. Hille, "Mathematics and Mathematicians from Abel to Zermelo," *The Mathematics Magazine*, XXVI (January-February, 1953), 143.

for the teacher to give, by a justification of the terminology or at least by a historical allusion, some flesh and blood to what otherwise might be regarded by some students as just some more of the dead bones of mathematics.

Then there are the misnomers in the more restricted sense of that term. Fortunately these are comparatively rare in mathematics. For the most part mathematical concepts are well named, meaningfully named. This is true largely because mathematical concepts, being technical and abstract, were named, in general, by mathematicians themselves, and not by the scientifically untrained, as would seem to be the case in the origin of the popular names of many plants, animals, inventions, and practical techniques. One will not find in a mathematician's bill of fare many analogues of the "Kitchen Delutions" mentioned in Leonard A. Paris' iconoclastic verse:

> The pineapple is no apple;
> The grapefruit bears no grape;
> The eggplant never hatched a chick;
> The apricot, no ape.
> All these are false and sinful,
> But here's what makes me glower:
> The lie that's foisted on the nose
> By so-called cauli*flower!*[8]

Indeed, most of the misnomers in the mathematician's vocabulary are not to be attributed to mistaken identity at all, but rather to the vagaries that some words go through in the course of time. It is a favorite field of investigation among all those who are interested in the etymology of words to search out how words that have come to denote widely different concepts can be traced back to the same source. Examples of this would be the terms *irrational*[9] and *witch* (curve of Agnesi).[10] And there are a few misnomers, curiously connected with mathematics, yet hardly to be regarded as mathematical terms at all. Their history, however, is invariably interesting and worthy of a brief reference in the course of our teaching. The word *love*, for example, which one hears bandied back and forth on tennis courts in such expressions as *thirty-love* and *forty-love*, has nothing to do with amorous sentiments, as anyone who has played strenuous competitive tennis knows. This expression is but the English corruption of the French *l'oeuf* —the egg, which in the context means zero. The English expression "goose egg" meaning zero is an example of the same usage.

Unfortunately there are examples of terms which derive from the same origin and nevertheless denote incompatible mathematical concepts. Note, for example, the various meanings of *mille* [L. *mille*—thousand]. The prefix *milli-* generally denotes one-thousandth part of, as in *milli*meter, but occasionally it means a thousand times, as in *milli*fold. Also, *million* denotes 1,000,000 but *billion* [L. *bis* —twice, million] means variously a thousand millions and also a million millions. In the matter of agreement on such terminology, especially between nations, there is much work to be done. Again, note the different uses of *cent* [L. *centum*—hundred]. We speak of *per cent* [L. *per*—through, by; *centum*—hundred]—division by a hundred. (Be sure to mention to your classes that our per cent symbol % developed by natural stages from an earlier symbol much like "Pc," somewhat as did the division symbol ÷.[11]) But our *cent* symbol ¢ is a contraction for the Latin word *centum:* centum→cent→ct→¢, and we note that the value of the original Latin "centum" has shrunk too, decimated and then decimated again, into our "cent" (a hundredth part of the dollar).

[8] Reprinted by special permission of *The Saturday Evening Post*. Copyright 1953 by Curtis Publishing Co.

[9] Louis E. Diamond, "Irrational Numbers," *The Mathematics Magazine*, XXIX (November-December, 1955), 97; Fred G. Elston, "Comment on Mr. Diamond's 'Irrational Numbers,'" *The Mathematics Magazine*, XXIX (March–April, 1956), 206–7.

[10] T. F. Mulcrone, S.J., "The Names of the Curve of Agnesi," *The American Mathematical Monthly*, LXIV (May, 1957), 359–61.

[11] Florian Cajori, *A History of Mathematical Notation* (Chicago: Open Court Publishing Co., 1928), I, 312–13.

The evolution of the symbols %, ÷, and ¢ are examples of the way in which very many of our symbols assumed their present form. Additional examples which are closely related to those just mentioned are the symbols $\sum$ and $\int$ which are variants of the letter *s* representing sum; and our dollar mark $ is supposed to have had its origin in the superimposing of the letters *U* and *S* representing United States. We should eagerly search out for ourselves and share with our students the origin, or the better conjectures as to the origin, of as many of our mathematical symbols as we possibly can. Nor should this solicitude in explaining terminology cease when we are teaching more advanced subject matter. It is easy for veteran teachers to introduce such terminology as $\sin x$, $\arcsin x$, $\sin^{-1} x$, and dy/dx without explanation and motivation, forgetful that they thereby contribute to the lack of enthusiasm and often enough the bewilderment of many of their students.

To your zealous and devoted interest then, I commend anew the world of words, a wonderous and enriching world of which, I think, we should all make much more account. There is something very solemn in having an instrument like language placed in our hands. We have a duty to see that our words, the mathematician's terms, are not allowed to become on our lips trite, dead syllables. Our words are alive, our words are pregnant. They are for us both the trophies of our intellectual past and the weapons of our future progress. They are for us the precious links whereby mathematics is not only singularly conversant within itself, but also with present physical reality and with our cultural past.

BIBLIOGRAPHY

BLAKELY, BERNICE. "In What Ways Can the Mathematics Teacher Develop a Meaningful Mathematics Vocabulary and Not Destroy the Meaning Those Same Words May Have in Outside Situations," *School Science and Mathematics*, XLVI (May, 1946), 463–67.

BRUNE, I. H. "Let's Look at Language," THE MATHEMATICS TEACHER, XLIX (January, February, April, May, 1956), 43, 153, 277, 365.

JONES, PHILLIP S. "Historically Speaking, Word Origins," THE MATHEMATICS TEACHER, XLVII (March, 1954), 195–96.

KEESLER, EARL R., "Vocabulary in Plane Geometry," THE MATHEMATICS TEACHER, XXXIV (November, 1942), 331.

SIDDONS, A. W. "In Defense of the Word 'Isosceles'," *Mathematical Gazette*, XXXIII (1949), 28. Reprinted in *Scripta Mathematica*, XV (1949), 237.

SMITH, D. E. "Changes in Elementary Mathematical Terms in the Last Three Centuries," *Scripta Mathematica*, III (1935), 291–300.

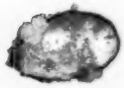
Quotations from Bertrand Russell

"The infinite furnishes an interesting sport for the mathematician. The question whether infinite collections exist is still debated among mathematicians. But the mathematical concept of infinity tells us nothing about the finitude or infinity of the world. That question must be settled on empirical grounds. It is not an *a priori* matter, as Kant supposed. Geometry is not concerned with space as Kant, who thought that Euclidean geometry determines space, supposed.

"To quote Russell: 'It was formerly supposed that Geometry was the study of the nature of the space in which we live. . . . But it has gradually appeared, by the increase of non-Euclidean systems, that Geometry throws no more light upon the nature of space than Arithmetic upon the population of the United States.'

"Mathematics is a fascinating game of logic, as Russell has shown; and it can be made an instrument of research, but only by selection. It does not dictate to reality. Metaphysics must liberate itself from mathematics, or, rather, from mathematicizing philosophers, who confuse logic with metaphysics."—*Taken from* The Philosophy of Bertrand Russell (*Evanston: The Library of Living Philosophers, Inc., 1946*), *pp. 479–80.*

The trigonometric functions[1]



ROBERT C. YATES, *College of William and Mary, Williamsburg, Virginia.*

Herein is presented a definition of trigonometric functions not usually found in elementary texts. Continuing in the same unusual way, properties of these functions are derived in a manner simple enough to use in an early calculus course.

THE TRIGONOMETRIC (circular) functions are basically and most satisfactorily defined by power series. The calculus student understands how these series are used in the calculation of function values for the construction of tables. The purpose of this paper is to show, however, that these series completely define the functions (that is, all properties are forthcoming) and to recommend a similar presentation at the end of the customary sophomore course. Of particular interest are two matters: the development of the sine and cosine of the sum, and the discovery of the periodic character of these functions (i.e., that $f(x+P)=f(x)$).

It is, of course, necessary to assume that the term-by-term differentiation of a series that defines a function produces a new series that defines the derivative of the function at least within the same interval.

The impact of the evolution here of familiar properties is best appreciated if the reader will pretend ignorance of these well-known functions.

1. *Definition of the function* S. Let S be the function defined by

$$S(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots, \qquad -\infty<x<\infty$$

and note that $S(0)=0$. Then, denoting the derived function $S'(x)$ by the symbol C:

$$S'(x)=C(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots,$$

$$-\infty<x<-\infty,\ C(0)=1.$$

We discover on further differentiation that

$$C'(x)=-x+\frac{x^3}{3!}-\cdots=-S(x)$$

or

$$S''(x)=-S(x).$$

Now

$$S''=\frac{dS'}{dx}=\frac{dC}{dx}=\frac{dC}{dS}\cdot\frac{dS}{dx}=\frac{dC}{dS}\cdot C.$$

The last result may then be written as

$$C\,dC=-S\,dS$$

and integrated to give

$$\frac{C^2(x)}{2}+\frac{S^2(x)}{2}=\text{Constant}.$$

But for $x=0$, $C(0)=1$, $S(0)=0$, and the constant is thus evaluated as 1/2. Accordingly

$$C^2(x)+S^2(x)=1, \qquad (1)$$

the fundamental identity.

[1] This is a companion article to Robert C. Yates, "The Logarithm and Its Inverse," THE MATHEMATICS TEACHER, LI (February, 1958), 105–6.

2. *The addition theorem.* Consider the determinant[2]

$$\begin{vmatrix} S(a+x) & C(x) & S(x) \\ C(a+x) & -S(x) & C(x) \\ -S(a+x) & -C(x) & -S(x) \end{vmatrix}.$$

Since the last row is the negative of the first, the determinant is zero for all a and x. Moreover, since the cofactor of $S(a+x)$ is not zero, we may expand in terms of elements in the first row. Thus

$$S(a+x)=\lambda\cdot C(x)+\mu\cdot S(x)$$

where

$$\lambda=-S(x)C(a+x)+S(a+x)C(x)$$
$$\mu=\ C(x)C(a+x)+S(a+x)S(x).$$

Their derivatives:

$$\lambda'=\ S(x)S(a+x)-C(x)C(a+x)$$
$$-S(a+x)S(x)+C(a+x)C(x)$$
$$\mu'=-C(x)S(a+x)-S(x)C(a+x)$$
$$+S(a+x)C(x)+C(a+x)S(x)$$

are identically zero. Thus each is a constant (independent of x), and their values may be determined by setting $x=0$. We have

$$\lambda=S(a),\qquad \mu=C(a).$$

Accordingly, for $S(a+x)$ and its derivative $C(a+x)$:

$$\boxed{\begin{aligned} S(a+x)&=S(a)\cdot C(x)+C(a)\cdot S(x)\\ C(a+x)&=C(a)\cdot C(x)-S(a)\cdot S(x)\end{aligned}}\quad(2)$$

3. *Zeros of* C(x). The function $C(x)$ is zero somewhere in the interval (1, 2). For

$$C(1)=1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots$$

is a convergent alternating series with *positive sum* less than 1. Furthermore, for similar reasons,

$$C(2)=1-2+\frac{2^4}{4!}-\frac{2^6}{6!}+\cdots \text{ is } \textit{negative}.$$

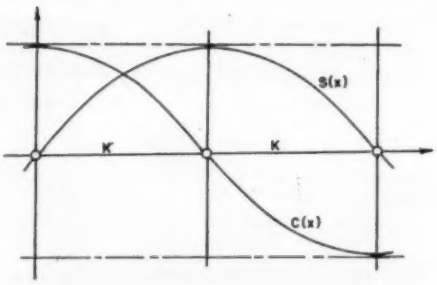


Figure 1

Since $C(x)$ is continuous, and $C(1)>0$, $C(2)<0$, $C(x)$ thus is zero at $x=K$, where

$$1<K<2.$$

Then

$$C(K)=0,\quad S(K)=1.\qquad \text{by (1)}$$

The derivative of $C(x)$:

$$C'(x)=-x\left(1-\frac{x^2}{3!}\right)-\frac{x^5}{5!}\cdot\left(1-\frac{x^2}{6\cdot 7}\right)-\cdots$$

is recognizably negative for all x in (1,2) and $C(x)$ is therefore a decreasing function *throughout* the interval. Thus $x=K$ is the only zero of $C(x)$ in (1,2).

4. *Periodicity.* The addition formulas now lead *pari passu* to values of $S(nK)$ and $C(nK)$ for integral n. Thus

for $a=K$, $x=K$:
$S(2K)=0$, $C(2K)=-1$;
for $a=2K$, $x=K$:
$S(3K)=-1$, $C(3K)=0$;
for $a=2K$, $x=2K$:
$S(4K)=0$, $C(4K)=1$.

A further inspection of the addition for-

[2] Admittedly, this is one of those things pulled down out of thin air for a special purpose and to many may seem but a low form of cunning. However, it may be recognized as the Wronskian of $S(a+x)$, $C(x)$, and $S(x)$, a good device for examining linear dependence.

mulas indicates, since $C(x)$ is *even*, $S(x)$ *odd*, that

$$S(K+x)=S(K-x)=C(x).$$

This is a *translation theorem:* graphs of $S(x)$ and $C(x)$ are congruent, one "leading" the other by an x-interval K. Moreover, $S(x)$ is symmetrical about $x=K$.

Results obtained in the preceding paragraphs establish periodicity. For, inserting values just obtained in the addition formulas (2):

$$\boxed{\begin{aligned} S(4K+x)&=S(x)\\ C(4K+x)&=C(x)\end{aligned}}.$$

Thus $S(x)$ and $C(x)$ are periodic with period $4K$.

To evaluate K, consider the circle

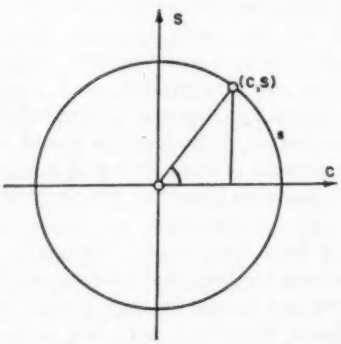


Figure 2

which by elementary geometry has area π. By the calculus, the quarter area is

$$\frac{\pi}{4}=-\int_{C=0}^{C=1} S\,dC=\int_{S=0}^{S=1} C\,dS,$$

the two integrals being equal by virtue of the symmetry of S and C in $C^2+S^2=1$. To convert to the variable x:

$$dS=C(x)dx, \qquad dC=-S(x)dx.$$

With appropriate limits of integration:

$x=K$ corresponds to $C=0$ and to $S=1$,

$x=0$ corresponds to $C=1$ and to $S=0$,

there follows,

$$\frac{\pi}{4}=\int_{K}^{0} -S^2(x)dx=\int_{0}^{K} C^2(x)dx,$$

or, adding,

$$\int_{0}^{K}\left[S^2(x)+C^2(x)\right]\cdot dx=\frac{\pi}{2}\cdot$$

The bracketed expression is 1. Accordingly,

$$K=\frac{\pi}{2},$$

and thus $S(x)$ and $C(x)$ each has period $4K=2\pi$.

The remaining trigonometric functions are now defined, as is the custom, as algebraic functions of either $S(x)$ or $C(x)$. The whole of analytical trigonometry is now available.

5. *Angles.* To bridge the gap to the elementary definitions of $S(x)$ and $C(x)$ as functions of an angle measured in radians, we calculate the arc length of the circle (Fig. 2):

$$S^2+C^2=1.$$

from the C-axis to the point (C, S).

Since C here will play the role of independent variable, arc length s is given by

$$s=-\int_{1}^{C}\sqrt{1-\left(\frac{dS}{dC}\right)^2}\cdot dC.$$

Now

$$\begin{aligned} dS&=C(x)dx\\ dC&=-S(x)dx\end{aligned} \quad \text{and} \quad x=0 \text{ for } C=1.$$

Thus

$$s=-\int_{0}^{x}\sqrt{1+\left(\frac{C}{-S}\right)^2}\cdot(-S\,dx)=x.$$

Therefore, since the arc of a unit circle measures the angle subtended, x is that angle.

Observations on the teaching of mathematics in the United States

CALEB GATTEGNO,* *UNESCO Technical Assistant, Addis Ababa, Ethiopia.*

A visitor to the United States sets forth his impressions of some current events in mathematics teaching and enumerates some important problems in mathematical instruction.

TO THE OBSERVER, the American scene offers many points of comparison with other countries, and although in the short time I spent in the United States it was not possible to get a complete picture, I venture to express an opinion in the hope that something useful may emerge from my examination, particularly as my experience of a number of countries enables me to be more receptive to differences and salient points.

Since I am writing mainly for American teachers, I am assuming that the setting is familiar. I had the good fortune to spend most of the eight weeks of my visit in New York, Chicago, and the San Francisco area. I spent much time in libraries and in talking to teachers at various levels from kindergarten to university.

My remarks fall under five headings, and, although they are related, they can be viewed independently since they refer to different people and different areas of education.

TEACHERS AND TEACHERS' MANUALS

There are few parts of the world in which schoolteachers are held in high esteem by the public, but nowhere have I found them professionally so little trusted as in the United States. This may be an exaggerated view of the situation, but it derives from what I gathered at both ends: from the teachers themselves and from the educational authorities. The latter, feeling that the teachers are not interested, are not ready to make efforts to raise the status of their teachers. These authorities are ill-prepared, and consider that radical and urgently needed reform is impossible since the teachers would resist it on account of their feeling of inadequacy. If this were indeed so, the problem being well circumscribed, an attack on it should have been made. Personally, my experience has been that when teachers are trusted and helped to grow toward independence, they work hard and overcome the obstacles.

In seeking an explanation for such a state of affairs, I found the teacher's manual, an institution which is common in the U.S.A. but which I have not met elsewhere. All the reasons I was given for the existence of the manual hinged on the idea of the inadequacy of the teacher confronted with his job. Nothing was said of the damage done to the teacher's self-respect and to his opportunities for initiative. To my mind, a textbook that is placed in the hands of the pupils can display its philosophy and the way it can be used equally well to the child as to the teacher. To assume that teachers are lazy, busy,

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and ill-equipped is to support my contention, but are U.S. teachers in fact worse than any others in the world? Having met a few hundred teachers during my visit, I do not believe it—or did I, by pure chance, meet only the cream of the profession? There was, moreover, no suggestion that the manual has solved any teaching problem, but merely complaints as to the low level of teaching, which was compared adversely with that of European teachers, who are never subjected to this drastic treatment.

Educators and complacency

My second point is that a certain complacency is to be found in those concerned with the teaching of mathematics. Let me explain myself.

In looking at the literature and listening to what is said, one sees that the public wants improvement *now* and is showing its impatience by readiness to increase teachers' salaries, whereas those in authority in colleges and universities are thinking in terms of decades, which means that some kind of miracle would be needed to meet the situation.

Although scientific research is desirable whenever possible, there is much presentation in education of evidence which is not relevant to the challenges, and conclusions are drawn which involve children and teachers and which are far from being legitimate. Most of my reading in the libraries to which I had access indicated that there was much more concern for the opinions of the leaders than for the problems facing us. Much of what is expressed in experimental language and in the jargon that is fashionable is trivial or irrelevant. This research drive, which on the whole is yielding little fruit, needs to be seriously examined.

When I inquired of those leaders I met what effect research was having on teaching techniques, there was no satisfactory answer. Two positive assertions were made, and these derive from belief rather than from research: (1) that modern mathematics must become part of the high school program and that teachers must be instructed in its content; (2) that there should be less social arithmetic and more mathematics as such in the elementary school. All the other changes are concerned with detail of exposition; no frontal attack is contemplated, although so many people are working in the field.

Mathematicians and the reform

The work being done in order to introduce modern mathematics into the secondary schools differs in many ways from what is being done in Europe, and this work seems to me to lack "seriousness." It is, in fact, mathematicians as such who are given a free hand in the changes proposed, and as they know only one of the components of the teaching situation, they can easily mistake a neat approach for the right one. In particular, their stress on rigor and on logical sequences leads almost inevitably to the postponing of the introduction of powerful but elementary ideas until a very late date in the school life of the population. If the mathematicians' sense of rigor could operate equally well in the pedagogical context, they would have undertaken a prior indispensable study: "What can children understand of the modern ideas of mathematics, and at what age?" It does not appear that many of those writing on the topic have much idea as to the answer. Having considered that it was my responsibility as an educator to investigate this question, I found in fact that modern mathematics can be started right at the beginning of the elementary school and a thorough job made of it from all points of view.

Instruction and learning

The fourth remark I want to make concerns the emphasis on *instruction* rather than on learning. Although most theories of learning originated in America, the relation of teacher to pupil is in terms of instruction, and as this seems to be generally

unquestioned, I assume that it is not just a passing fashion. To my mind, this stress on instruction is part of the unconscious ideology of the people of the United States, who believe in saving time and consider that if one is told something true there is no need to waste time in finding out about it. But can we find justification for it when we see the heavy price children pay for so few of them to be effectively instructed?

This unconscious confusion of learning with instruction has moreover another consequence of import. It is said that children's learning is being studied, but this is not what actually takes place. What occurs instead is the inclusion in their activity of the various factors brought by the teachers, by the current ideas on child development, etc. For example, the legitimate question: "How do children learn to subtract two numbers?" becomes in fact the following: "Given that we present children with two numbers written in the decimal notation, and given that we place the two numbers in such a way that the units, the tens, etc. are related in particular ways, can children grasp what we tell them to do? And which is the most efficient device for the purpose?" This, to me, is not a study of children and subtraction. It would be interesting to know the reader's reaction to the following statement: "In my work, I have found that children do not need to learn the procedure of subtraction advocated for centuries, and that children make far fewer mistakes by substituting for the given subtraction an equivalent one that they can solve at once." It was by studying subtraction, not as a procedure, but in relation to the initiative of the child, that I made this unexpected discovery.

The scientific approach and teaching

Very few people are aware that children, ordinary children, can deal with mathematical situations of a much more elaborate nature than those with which they are usually presented. The reason for this is that educational psychologists, for mysterious reasons, are interested in percentages of success and failure, in time taken during tests, etc., and these psychologists find no time for the direct study of what needs study. Too busy being "objective," they empty situations of what seems to matter, and then find this or that. What is needed today is a renewal of ourselves so that we do not break up the challenges into such small units that we kill the problem. The scientific approach is not identical with annihilation. To be scientific in the field of mathematics teaching, it is essential that we see the field, and that we meet the problems in their proper setting; and this goal requires the telescope as well as the microscope.

Whenever we forget that the learning of mathematics is concerned with people, with mathematical structures, and with dialogues of persons about a mental activity, we are in danger of reaching false conclusions. I cannot say that many of those who write or talk about mathematics teaching seem to me to be continuously taking all the factors into account.

Conclusion

The foregoing remarks are made in the light of experience gained through working in conditions which may not have been the most favorable. As a mathematician interested in the improvement of teaching, I find that this is the best way of expressing my thanks to all those who helped me gain this experience.

In view of comparatively recent comments on the meaning of "=," this quotation taken from G. Chrystal's *Algebra: An Elementary Textbook* (1886) is of interest. "It (the sign '=') means *is transformable into* by applying the laws of algebra, without any assumption regarding the operands involved."

The student stockholders

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A group of New Hyde Park, New York students learn about stocks and stock markets through the actual purchase of one share of stock.

WHEN I WAS PLANNING for my ninth-year general mathematics course this year, I spent a considerable amount of time trying to work out plans for a unit of work that would have real meaning for my students. One day, my chance arrived.

It all began at the Herricks Junior High School, in New Hyde Park, New York, as the class was discussing the work that was to be covered for the year and trying to select a topic that they would like to investigate for our next area of concentration.

Upon examination of the possibilities as presented in the New York State syllabus, most of the youngsters seemed especially interested in an item listed simply as "Stocks and Bonds." After due consideration, eliminating other possibilities, the stocks topic was selected as the next topic for investigation. "How shall we go about this?" I asked the class. During the ensuing exchange of ideas, I recalled an item I had read about a project carried out at Nyack High School, where a class had actually purchased a share of stock as part of a school unit. I threw this idea out to the group for consideration. Although the idea met with much enthusiasm, the students were rather skeptical about such an activity ever taking place. When I assured them it could be done if they so desired, they proceeded to discuss arrangements. We went ahead considering all possibilities, and after several planning sessions, we had together planned the unit's work.

In the meantime, I had explained this activity to our principal, Wilbur Olmstead. He thought the idea was a good one and suggested that a letter be sent to each student's home explaining our plans and seeking out parental reaction and suggestions. When the group had made a definite outline of its plans, the task of writing the letters to parents was made a group project and carried out co-operatively.

After working out the text of the letter, a student typed the copy on a ditto stencil, and enough copies were run off for all the members of our group. The letters were sent home, and the parental reaction was enthusiastically, and unanimously, in favor of our plans for the actual stock purchase. We had the green light and went ahead.

The group became concerned about the eventual disposition of the stock and chose to decide this question first. With much seriousness, and only after a lengthy but careful exchange of ideas, the students decided to have the stock sold sometime near the end of May. All the proceeds received, up to the original investment, were to be donated to a charity to be selected later. Any profits that might accrue should be distributed to the shareholders. Our attention then turned to the problem of selecting a good investment purchase. Close examination of the *New York Times* financial section brought a sharp awareness to the students of the difficulty of our task. There were thousands of stocks from

which to make our selection. To aid us in our selection, the role of the major exchanges in New York City were discussed and a field trip arranged to visit the New York Stock Exchange. Our experience with the New York Exchange brought out the numerous requirements to be met before a stock is accepted for listing. This impressed the class to the point where they decided that their purchase would be made by considering only those stocks listed on the "Big Board." We had narrowed the field, but there were still some 1200 corporation issues to be considered.

At this point I suggested that we take time to consider the amount the class would invest. The group indicated that they would like to limit the size of their investment to about fifty cents each. With twenty-three students participating, our total investment came to $11.50, and automatically eliminated any stock costing as much as $15.00 per share. Our task of weeding out had begun.

A quick examination of the stock list yielded many stocks in our price range, and a suggestion was made and accepted by the group to the effect that we eliminate all stocks that did not pay a dividend. To prevent the overburdening of any one group member, the class divided the stock list by alphabetical arrangement, with each student being responsible for selecting from his assigned section all listed stocks under $15.00 per share that paid dividends. The first tentative list for selection had been made.

I would like to describe what had been learned as a natural result of bringing the unit to the indicated stage of development. I found the actual benefits to the class to be of two major kinds. First, there was the progress that had been made in the content areas. Progress here had been achieved because the children needed to know:

1. Information about corporate structure.
2. The requirements to be met before securities may be publicly sold.
3. The role of the stockbroker.
4. The many kinds of securities available for public investment, and the need that each type tries to fulfill.
5. Other possible avenues of investment besides corporate securities.
6. All the many items to be considered before money is invested.

The preceding list indicates some of our more important achievements under the heading of content.

A second kind of learning area in which progress was made was the area of social living and effective group functioning. The values derived here are many:

1. The ability was developed to carry on discussions in such a way that real sharing of ideas took place.
2. The ability to state ideas clearly was developed.
3. The responsibility for planning and implementing plans was shared by all group members.
4. Respect for fellow group members was implicit in everything that was done.
5. Respect for oneself was developed by each member's role in group actions.

These goals are more difficult to achieve than is proficiency in content. I pretend no yardstick to measure progress in the above areas. I simply present my impressions from my own role as a participant in the group.

The development of the unit went forward. A preliminary list was constructed of stocks that were listed on the New York Stock Exchange, and were selling at $15.00 per share or less and paying dividends.

Although many stocks had been eliminated, the remaining list was sizable. Ideas were called for to trim the list even further. The group elected to compute the yield (per cent of dividend) of each stock and compare the results, deleting those stocks of low yield. Since the list was lengthy, the group once more turned to a sharing of this work, with the computations equally divided among the members of the class. Those students unable to carry out this computation gathered at the blackboard and went over the process in-

volved, a process taught and eagerly accepted since the need was real.

All the stocks bearing a 5 per cent yield or better made up a final list of twenty stocks for careful scrutiny. It was at this point that I contacted Merrill, Lynch, Pierce, Fenner and Beane, and Jeremiah Murphy of that firm listened to my explanation of our plan. His help, consideration, and patience made our success with this unit possible. He forwarded to me fact sheets on each of the twenty corporations the class had selected, together with endless supplies of informational pamphlets enabling me to place into the hands of each child, material that was considered by the group. Such publications as "How to Read a Financial Statement," "How to Invest," "Stocks Paying Dividends Consistently," and "111 Growth Stocks" are only a few of the many that were sent to us. These publications proved invaluable in helping us complete our analysis.

Upon receipt of the twenty requested fact sheets, the class elected to assign the stocks at random and to accept the responsibility for their analysis and a recommendation as to possible purchase. Before beginning this work, the group discussed the question of, "What is it that I want to know about the stock in order to help me decide upon its value as a possible investment?" A list of important questions was compiled to help guide each student's analysis. This list was duplicated so that each member of the group got a copy. A time schedule was set up for reporting back to the group, and work sheets were devised to facilitate the recording of information from the oral reports, and the students proceeded.

It was during this phase of development that a great many mathematical learnings took place. The question of ratio arose; the per cent of profit had to be computed; graphs had to be constructed. These and many other computational requirements made necessary several lessons based entirely on content. Whenever the needs of my students indicated such lessons were necessary, the essential instruction was given.

The reports were made. The reports involved: the answering of all questions previously listed by the group for consideration by all; a personal recommendation of the student reporting, with full explanations; and a question-and-answer session to consider any points that were in need of clarification. These reports consumed the better part of two weeks. On the form the class devised for recording of information, a space was provided for a "buy" or "don't buy" recommendation by each student for every stock on our list. When all the oral reports were presented, each student was allowed one minute to give an over-all summary of his stock and restate the recommendation of further consideration or the dropping of the stock from our list. This concluded the presentations and brought about a group examination of each stock in turn. The group chose to scrutinize first those stocks receiving adverse recommendations from the reporting student. Generally speaking, these issues were the first stocks to be eliminated. The group established a rule of thumb that whenever five or more people declared against a particular stock, that stock be dropped from further consideration. In this manner the field was eventually narrowed down to four stocks: A. B. C. Vending, Minute Maid, Vertientes Sugar, and Barker Brothers. A decision was difficult. The group finally decided that they would help themselves to decide by re-examining our literature in an attempt to seek out criteria that may not have been given careful consideration. This information would help the group to extend their comparison. One outcome of this project was the computation of the price-earning ratio for each stock. This new evidence was a big factor in the decision-making and finally left the group with a choice between A. B. C. Vending and Minute Maid. After further examination the final choice was Minute Maid.

That same day the order was phoned in to Mr. Murphy at Merrill Lynch, Pierce, Fenner and Beane, and our class was in business.

How would we keep track of our investment? Again, after much group discussion, it was agreed to have the students take turns checking the *New York Times* each day. The price would be placed on the blackboard for everyone to see. Each Friday's price would be recorded by everyone in the class in a table, and a running line graph would be kept by each student to trace changes in price. In addition a weekly bar graph would be made of the total number of shares sold each week. The class also accepted a suggestion that I made to the effect that five minutes be taken each Monday from our mathematics class to discuss the progress made during the previous week and possible causes of any notable changes in price. These ideas provided for a continuous follow-up.

In February, we received notice of a stockholders meeting, which would make an excellent class trip. I immediately set about making arrangements for our group to attend. To my dismay there was limited space provided for this meeting. This led the group to select five people to represent them at the meeting and report back to the class all that had taken place. Not only were we introduced by the president of the corporation to the people present, but we were also given seats in the front of the room. The students were thrilled. Their reports back to the group were eagerly awaited and well presented. It was a rich and full experience.

There you have a brief sketch of our work to date. Please understand that a chronological report like this tends to smooth out rough spots (there were many) and gloss over several highlights (also numerous). I present this report as one attempt to teach by experience. If it provides someone with a stimulus to experiment, the report will have been as valuable to the reader as it was to my class. The students developed mathematical skills; the ability to analyze, compile, and present data; the ability to accept responsibility and work successfully in a group; and much informational value in relation to investment. Could I expect more?

Quality of teaching

Teachers College, Columbia University, asked 1000 high school teachers and administrators what they saw as the reasons behind current science enrollment trends in our high schools. More than 600 of them responded within three weeks, and a summary of the results shows that the quality of teaching led all the other reasons given for science enrollment changes by a very wide margin. In those schools where such enrollments are increasing, good teaching was given the credit, although auxiliary reasons were mentioned. Among them were advice from well-staffed and well-prepared guidance counselors, favorable home environment, national publicity on the need for scientifically trained personnel, and good opportunities in industry.

For the schools where science enrollments are declining, the responsibility was placed primarily on poorly prepared or uninteresting teachers. Supplementary reasons given were, of course, the alleged difficulty of the subject, its conflict with extracurricular activities, and inadequate guidance facilities. In general it is the teacher who holds the key to the situation.

Without benefit of a survey we might extend the conclusion to the college level. Many a student with no intention of majoring in science has been drawn into a scientific or engineering career by good teaching. Conversely, many an enthusiastic high school boy or girl, bent on making science or engineering a career, has been repelled, if not by poorly prepared teachers, certainly by uninteresting ones.—*Taken from* Newsletter, *#104, April 18, 1957* (*Engineering and Scientific Manpower*).

• HISTORICALLY SPEAKING,—

Edited by Howard W. Eves, University of Maine, Orono, Maine

*The prime numbers**

With this issue, Howard Eves will assume the editorship of Historically Speaking. Because of the pressure of numerous activities, Phillip Jones has found it necessary to resign as editor of this department. We owe much to Phillip Jones for his efforts and the contributions he has made to the magazine.—*The Editor.*

Proposition 14 of Book IX of Euclid's *Elements* is essentially the equivalent of the important "fundamental theorem of arithmetic," which states that *any integer greater than 1 can, except for the order of the factors, be expressed as a product of primes in one and only one way.* This theorem asserts that the prime numbers are building bricks from which all other integers may be made. Accordingly, the prime numbers have received much study, and considerable efforts have been spent trying to determine the nature of their distribution in the sequence of positive integers. The chief results along this line obtained in antiquity are Euclid's proof of the infinitude of primes and Eratosthenes' *sieve* for finding all primes below a given integer n.

Euclid's proof, in Proposition 20 of Book IX of his *Elements*, that *the number of prime numbers is infinite*, has been universally regarded by mathematicians as a model of mathematical elegance. The proof employs the indirect method, or *reductio ad absurdum*, and runs essentially as follows. Suppose there is only a finite number of prime numbers, which we shall denote by $a, b, \cdots, k$. Set $P=ab\cdots k$. Then $P+1$ is either prime or composite. But, since $a, b, \cdots, k$ are all the primes, $P+1$, which is greater than each of $a, b, \cdots, k$, cannot be a prime. On the other hand, if $P+1$ is composite, it must be divisible by some prime p. But p must be a member of the set $a, b, \cdots, k$ of all primes, which means that p is a divisor of P. Consequently, p cannot divide $P+1$, since $p>1$. Thus our initial hypothesis that the number of primes is finite is untenable, and the theorem is established.

The so-called *sieve of Eratosthenes* is a clever device noted by the Greek mathematician Eratosthenes (*ca.* 230 B.C.) for finding all the prime numbers less than a given number n. One writes down, in order and starting with 3, all the odd numbers less than n. The composite numbers in the sequence are then sifted out by crossing off, from 3, every third number, then from the next remaining number, 5, every fifth number, then from the next remaining number, 7, every seventh number, from the next remaining number, 11, every eleventh number, and so on. In the process some numbers will be crossed off more than once. All the remaining numbers, along with the number 2, constitute the list of primes less than n.

From the sieve of Eratosthenes can be obtained a cumbersome formula which will determine the number of primes below n when the primes below $\sqrt{n}$ are known. This formula was considerably improved

* Adapted from Howard Eves, *An Introduction to the History of Mathematics* (New York: Rinehart & Co., Inc., 1953).

in 1870 by Ernst Meissel, who succeeded in showing that the number of primes below 10^8 is 5,761,455. The Danish mathematician Bertelsen continued these computations and announced, in 1893, that the number of primes below 10^9 is 50,847,478. This represents our most extended knowledge along these lines.

No practicable procedure is yet known for testing large numbers for primality, and the effort spent on testing certain special numbers has been enormous. For more than 75 years the largest number actually verified as a prime was the 39-digit number

$$2^{127}-1 = 170,141,183,460,469,231,731,687,$$
$$303,715,884,105,727,$$

given by the French mathematician Anatole Lucas in 1876. In 1952, the EDSAC machine, in Cambridge, England, established the primality of the much larger (79-digit) number

$$180(2^{127}-1)^2+1,$$

and in the same year the SWAC digital computer, in the United States, established the primality of the enormous numbers $2^{521}-1$, $2^{607}-1$, and $2^{1279}-1$, the last of which is a 386-digit number.

A dream of number theorists is the finding of a function $f(n)$ which will yield prime numbers for all positive integral n. Thus $f(n)=n^2-n+41$ yields primes for all such $n<41$, but $f(41)=(41)^2$, a composite number. The quadratic polynomial $f(n)=n^2-79n+1601$ yields primes for all $n<80$. Polynomial functions can be obtained which will successively yield as many primes as desired, but no such function can be found which will always yield primes. It was about 1640 that the great number theorist, Pierre de Fermat, conjectured that $f(n)=2^{2^n}+1$ is prime for all nonnegative integral n. For $n=0, 1, 2, 3, 4$ we find $f(n)=3, 5, 17, 257, 65537$, all prime numbers, but in 1732 Euler proved the conjecture false by showing that $f(5)=(641)(6700417)$. It is now generally felt that $f(n)$ is composite for all other values of n, although this has not been established. An interesting recent result along these lines is the proof, by W. H. Mills in 1947, of the existence of a real number A such that the largest integer not exceeding A^{3^n} is a prime for every positive integer n. Nothing was shown about the actual value, nor even the rough magnitude, of the real number A.

A remarkable generalization of Euclid's theorem on the infinitude of the primes was established by Lejeune-Dirichlet (1805–59), who succeeded in showing that every arithmetic sequence,

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,$$

in which a and d are relatively prime, contains an infinitude of primes. The proof of this result is far from elementary.

Perhaps the most amazing result yet found concerning the distribution of the primes is the so-called *prime number theorem*. Suppose we let A_n denote the number of primes below n. The prime number theorem then says that $(A_n \log_e n)/n$ approaches 1 as n becomes larger and larger. In other words A_n/n, called the *density* of the primes among the first n integers, is approximated by $1/\log_e n$, the approximation improving as n increases. This theorem was conjectured by Gauss from an examination of a large table of primes, and was independently proved in 1896 by the French and Belgian mathematicians J. Hadamard and C. J. de la Vallée Poussin.

Extensive factor tables are valuable in researches on prime numbers. Such a table for all numbers up to 24,000 was published by J. H. Rahn in 1659, as an appendix to a book on algebra. In 1668, John Pell, of England, extended this table up to 100,000. As a result of appeals by the German mathematician J. H. Lambert, an extensive and ill-fated table was computed by a Viennese schoolmaster named Felkel. The first volume of Felkel's computations, giving factors of numbers up to 408,000, was published in 1776 at the expense of the Austrian imperial treasury. But there

were very few subscribers to the volume, and so the treasury recalled almost the entire edition and converted the paper into cartridges to be used in a war for killing Turks! In the nineteenth century, the combined efforts of Chernac, Burckhardt, Crelle, Glaisher, and the lightning calculator Dase led to a table covering all numbers up to 10,000,000, and published in ten volumes. The greatest achievement of this sort, however, is the table calculated by J. P. Kulik (1773–1863), at the University of Prague. His as yet unpublished manuscript is the result of a 20-year hobby, and covers all numbers up to 100,000,000. The best available factor table is that of the American mathematician D. N. Lehmer (1867–1938). It is a cleverly prepared one-volume table covering numbers up to 10,000,000.

There are many unproved conjectures regarding prime numbers. One of these is to the effect that there are infinitely many pairs of *twin primes*, or primes of the form p and $p+2$, like 3 and 5, 11 and 13, and 29 and 31. Another is the conjecture made by C. Goldbach in 1742 in a letter to Euler. Goldbach had observed that every even integer, except 2, seemed representable as the sum of two primes. Thus $4=2+2$, $6=3+3$, $8=5+3$, $\cdots$, $16=13+3$, $18=11+7$, $\cdots$, $48=29+19$, $\cdots$, $100=97+3$, and so forth. Progress on this problem was not made until 1931 when the Russian mathematician Schnirelmann showed that every positive integer can be represented as the sum of not more than 300,000 primes! Somewhat later the Russian mathematician Vinogradoff showed that there exists a positive integer N such that any integer $n>N$ can be expressed as the sum of at most four primes, but the proof in no way permits us to appraise the size of N.

Have you read?

MESERVE, BRUCE E. "A Teacher's View of Mathematics," *School Science and Mathematics*, December 1956, pp. 716–718.

Maybe this is just another view, but to me it seems more than that. For example, the author points out that when the pupil thinks of the symbol 5 he is abstracting and symbolizing. This provides the life of mathematics. When the concrete objects are discarded then the student relies on assumptions and properties of the number system. He obtains a feeling for sets of elements and their operations. Think of the set of all integers, of the points and lines in a plane, or of all complex numbers and you readily see the generalizations possible in mathematics. One of our goals must be to recognize the immensity and brilliance of the structure on which we are working. This requires re-evaluation and careful analysis because competence in the details is not enough. We must also recognize the rapid changes that are taking place in mathematics and its applications. This article is pleasant reading as well as inspiring.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

SALLEY, CHARLES M., AND MESSICK, SAMUEL J. "Probability, Learning, the Statistical Structure of Concepts, and Measurement of Meaning," *The American Journal of Psychology*, June 1957, pp. 161–173.

We have always been concerned about teaching meaning in mathematics. One of our real problems has been that of measuring the success of our teaching. The experiment carried out by the authors and reported in this article is an attempt to study a phase of this problem. They recognized that the concepts of an individual are the results of many variables. They considered the probability relationships that might exist. They used the semantic differential idea by asking the subjects to represent the concept on several different scales.

The procedure used was to let the subject view a lap card showing a tribe of stickmen and then view it again after being turned over for 5 seconds. Following this the subject was asked to rate it on a scale devised for this purpose. These results were then analyzed by several methods.

From this study, there seems to be some possibility of measuring meaning, but it must take into consideration the concepts that students have in relation to the meaning they are expected to acquire. It is not a simple process but such studies as this may help toward a solution of the problem.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

The stock market as a teaching device

by Douglas E. Scott, Amphitheater High School, Tucson, Arizona

The practical teaching unit in a high school general mathematics course should certainly involve more than one or two skills or processes, if possible. Ideally, too, practice in computation should be attained by use of techniques other than drill, since students, by the time they reach the ninth grade, often fail to profit from drill-type activities. In the February, 1957 issue of THE MATHEMATICS TEACHER, Harriet Schultz* described an application of the stock market on the junior high school level. Our general mathematics class was at that time just finishing the unit described here, which goes beyond the idea of observation, and involves actual participation in a stock market.

In constructing this unit, it was our desire to incorporate as many of the arithmetical skills as possible in addition to the basic material on the operation of a stock exchange and on the use of common stocks as an investment vehicle. As it finally developed, the unit provided practice in the fundamental computational processes, in percentage, in graph construction, and in accounting and record-keeping. At the end of the year the unit was rated by the class as the most interesting that they had studied.

Prior to the beginning of this unit, the class had studied investments and the general ways in which money may be invested. These ways were categorized into growth and nongrowth investments. Savings accounts, government and corporate bonds, insurance, and annuities were given as examples of nongrowth investments (in which there is no increase in the capital amount except as a result of compounding interest), while real estate and common stocks were used as examples of growth-type investments. The advantages and disadvantages of each type were discussed, and much emphasis was placed on the need for both types in a family investment program.

It might be well to mention here that in Amphitheater High School, some 50 per cent of the students in the ninth grade take a "business-training" course, in which these same ideas are very briefly outlined and then serve as a basis for purely computational problems on interest. It was partially in an attempt to avoid a duplication of material for the students enrolled in both business training and general mathematics that the idea of a classroom stock market was evolved. One student actually owned a share of stock in the local electric company, and several students brought in lists of stocks which their parents owned. Following the general discussion of ownership of a business through purchase of stock, we began following the performance of stocks of a few companies familiar to the students

* Harriet Schultz. "Junior High School and the Stock Market," THE MATHEMATICS TEACHER, L (February, 1957), 170.

(soft-drink companies, automobile manufacturers, etc.). Immediately, students began asking questions about the phenomena they were observing: "How are stocks sold? Could I buy stocks? What makes prices go up or down?"

We decided to set up our own stock market in an attempt to answer some of the questions that were being raised, but before doing so we thought it might be well to obtain some additional information. A representative of a local brokerage house with a seat on the New York Stock Exchange was invited to explain to us how that institution functioned. This talk was of great interest to the class, and the students became increasingly eager to see "just how all this stock business worked."

Each student was given $5000 to invest as he chose in stocks of five imaginary companies. One student in each row was designated as "broker" for his row—a different broker was assigned each day—with the task of recording, consolidating, and forwarding to the teacher the various "buy" and "sell" orders from his clients. The students kept two types of records: one, a record of cash income and expenditures; and the second, a record of what stocks were currently held, the dates of acquisition, number of shares, and total cost—with additional spaces for date of sale and amount received (if sold). At the end of each class period, every student prepared a "Daily Report," showing his beginning and ending cash balances, purchases, and sales (in both dollars and shares). This report was handed in, and served as one basis for grading during this unit.

The teacher's jobs were to (1) issue and redeem stock certificates, (2) record increases or decreases in market value according to demand, and, most important, (3) shape that demand by issuing "news releases" concerning anticipated increases or decreases in dividends, earnings, and so on. We had previously studied the relationships between dividends, earnings, and market value, and as it turned out, the students responded excellently to the cues given in the "news releases."

As each transaction was made, a new market value was computed by multiplying the present value by the ratio of the shares in the present transaction to the total number of shares outstanding. For instance, if the market value of "*X*" is 20 and there are 1000 shares outstanding, a purchase of 100 shares would raise the market two points. This is admittedly artificial but is admissible in the interests of simplicity and efficiency.

Our market operated for two weeks (ten school days) with Friday of each week being devoted to dividend distributions. At the end of the two-week period, each student handed in a summary of his activities in the market, which contained (1) a computation of the percentage yield produced by the dividends he received; (2) a computation of total worth, ascertaining whether he finished with more or less than his original $5000; and (3) a graph of the day-to-day performance of each of the stocks in his portfolio. This report, coupled with the daily reports, provided information for grading purposes. Points were awarded for accuracy of computation (which was extremely high) and for neatness—with some additional credit being given for good financial management as shown by a large increase in total worth.

The class did many things, then, during the total three weeks we spent on this unit. The students practiced computational skills with a purpose; they learned to keep accurate records of transactions; they applied their knowledge of percentage to "real" problems and made graphs of "real" numbers ("real" being used here as one student used it, as something distinct from "book" problems or numbers); they learned from lectures, outside speakers, and actual experience how stocks are bought and sold; and they discovered one of the reasons for the growth of America and its industry, and a way to share in that growth, if they wish, in the future.

A column of unofficial comment

It's up to us!

by George Immerzeel and Ross A. Nielsen, Campus High School, Iowa State Teachers College, Cedar Falls, Iowa

Recent world developments have put us, as teachers of junior high school mathematics, "on the spot." It is evident that the next few years will see many changes in the mathematics curriculum. What kind of changes shall they be? Shall we be satisfied with another rearrangement of the present program, or can we insist on something better?

Who shall be responsible for the changes that are to be made? Can we depend upon the leaders in the field of mathematics education and the publishing companies, or can we take an active part?

For over fifty years the leaders in the field of mathematics education have recognized the problem and have made suggestions for possible solutions. In 1902, in an address before the American Mathematical Society, E. H. Moore said:

> I understand that serious difficulties arise with children from nine to twelve years of age, who are no longer contented with the simple, concrete methods of earlier years, and who, never-the-less, are unable to appreciate the more abstract methods of the later years.[1]

Moore suggested as a partial solution to this problem:

> The children are to be taught to represent, according to the usual conventions, various formulas and interesting phenomena and to study the properties of the phenomena in the pictures: To know, for example, what concrete meaning attaches to the fact that a graph curve at a certain point is going down or going up or is horizontal. Thus the problems of percentage, interest, etc., have their depiction in straight or broken line graphs.[2]

In Germany in 1908, Felix Klein[3] indicated the importance of the transitional period when students are no longer primarily interested in extracting meaning from concrete objects, but rather are challenged by the abstract system of mathematics. He recommended that this transitional period emphasize the structure and continuity of the system.

In an article written for THE MATHEMATICS TEACHER in 1940, William Betz[4] stated that secondary school mathematics was plagued by a one-sided emphasis on "social utility"; by mechanistic, ineffective teaching methods; by a lack of clearly defined, convincing objectives; by inadequate textbooks; by a wrong psychology of learning; and by the tremendous problems of mass elevation.

From this small sampling of the think-

[1] Eliakim H. Moore, "On the Foundations of Mathematics," *A General Survey of Progress in the Last Twenty-five Years*, First Yearbook of the National Council of Teachers of Mathematics (Washington, D. C., 1926), p. 45.

[2] *Ibid.*, p. 46.

[3] Felix Klein, *Elementary Mathematics from an Advanced Standpoint: Arithmetic-Algebra-Analysis*, trans. E. R. Hedrick (New York: The Macmillan Company, 1932).

[4] William Betz, "The Present Situation in Secondary Mathematics," THE MATHEMATICS TEACHER, XXXIII (December, 1940), 339–60.

ing of the leaders of the past, it is evident that (1) the problem at the junior high school level has been recognized, and (2) possible solutions to this problem have been proposed.

Have the textbook materials reflected the thinking of these men? What changes have taken place in junior high school textbooks during this time? In comparing today's books with those of thirty years ago, we notice first the change in format. The general appearance and arrangement of materials is more appealing. Books contain a variety of color to emphasize pictures, diagrams, and even rules. The books have been written carefully to adapt the materials to the reading level of the pupils. Textbook problems are more realistic but are based upon "social utility" rather than the big ideas of mathematics.

As you compare the text materials of today with those of thirty years ago, you notice a different arrangement of the units, but the structure of mathematics and its abstract systems are not apparent to the student. According to W. D. Reeve, writing in 1929, the guiding principle in the selection of subject matter for the junior high school courses was "that the subject matter selected should be that material which will be most valuable to the pupil, provided he leaves school at the end of that year."[5] Even though a large majority of today's pupils can be expected to continue their education, these same terminal materials are to be found in present junior high school textbooks.

It is evident that much of the thinking of the leaders in mathematics education has not been reflected in the textbook materials available to the junior high school teacher. Can this be blamed on the leaders or publishers, or is it our fault as classroom teachers? The textbook companies can only be expected to publish the material which we teachers are ready and willing to use, regardless of what the leadership indicates should be done.

There are many indications today that certain aspects of our mathematics curriculum are inadequate. We need to realize that it is we, the classroom teachers, who must provide the impetus which is necessary to develop a modern program in mathematics. The leadership has been provided, the modern concepts of mathematics are being developed, and the publishers will produce the textbook materials when we as teachers indicate that we are ready to use them. It is up to us!

[5] W. D. Reeve, "United States," *Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910*, Fourth Yearbook of The National Council of Teachers of Mathematics (Washington, D. C., 1929), p. 166.

When do you think the following words were written? The answer, given in the inverted type below, may surprise you.

"Now the next problem is, how to teach pupils in classes in the graded system . . . and not sacrifice individual power and individual opportunity. Do you think that we have solved that question to our entire satisfaction? Is it not true that scores of bright boys and girls endowed with superior ability are chained down to the system . . . when if they had the liberty or power, they could accomplish the eight-year course in six?"

—NEA Proceedings, 1887.

—*Taken from* NEA Journal, *May 1957.*

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Basic Concepts of Mathematics, Karl Menger. Chicago, Illinois Institute of Technology Bookstore, 1957. Paper, vi+93 pp., $1.40.

This pamphlet should be studied by every teacher of mathematics—from those who introduce first graders to the beginnings of arithmetic, through those who direct graduate studies. For some time, Professor Menger has stressed the need for greater precision in the language used in mathematics, and he now attempts to show how this need can be satisfied insofar as the language of arithmetic and algebra is concerned. His attempt is sufficiently successful to justify, in the reviewer's mind, the emphatic recommendation which heads this review. However, after making such a recommendation, the reviewer would be remiss if he did not discuss at least some of the relatively few points in which he disagrees with the author. Most of the remainder of this review will be devoted to such a discussion.

Some of the more important disagreements refer to Chapter III, which deals with the use of variables in general statements. As an example, consider the statement (p. 30):

(1) $(x^2)^2 = x^4$ for any number x.

The author construes this as an abbreviation for:

(2) $(x^2)^2 = x^4$ where x may be replaced by the designation of any number.

Now the legend (to use the author's word) which follows the formula '$(x^2)^2 = x^4$' refers, in (1) to numbers, and in (2) to operations with numerals. Statement (1) is a true generalization about numbers, but (2) is nonsense. A statement which asserts what (2) attempts to assert is:

(3) The result of replacing 'x' in '$(x^2)^2 = x^4$' by any numeral is a true sentence.

Statement (3) is about expressions (not, as in (1), about numbers). It asserts that each statement which is an instance of (1) is true.

Statement (1) illustrates two unfortunate choices made systematically by the author—the choice of 'any' in preference to 'each', and the choice of using a legend as an affix rather than as a prefix. As to the first of these choices, the phrase 'for any x' does not serve the same purpose in the English language as does a universal quantifier, '(x)' or '$\forall_x$', in a formal language. And the English phrase which most nearly corresponds with the latter is 'for each x'. (Compare, for example, the false statement 'if for any x, $x>2$ then $0>2$' with the true statement 'if for each x, $x>2$ then $0>2$'.) As to the second choice, one has only to compare the author's examples:

$y>x$ for any x and some y (Ex 11″, p. 41)
and: $y>x$ for some y and any x
with: For each x, there is a y such that $y>x$
and: There is a y such that, for each x, $y>x$.

in order to realize that, in the interest of clarity, quantifying phrases should be prefixes.

Limitations of space preclude comments on the later chapters. But the reviewer cannot pass over the description of complex numbers (Chapter VIII) as "artificial." Whatever one's mathematical philosophy may be, surely the complex numbers are no more artificial than are the real numbers. And it is surely unkind to increase the teacher's burden by introducing another adjective which, like 'real' and 'imaginary', must be "explained away"!

Finally, the reader should be warned that there are numerous cases in which single quotes which should be present have been omitted. For example, this occurs twice in the note on page 7 which terminates paragraph 5. There are in fact fourteen such omissions on page 7. (Three of these may be justified by ruling that displayed expressions function as names for themselves.) And there may be two others, depending on whether 'for' stands for 'as an abbreviation for' or 'as a name for'.—*Herbert E. Vaughan, University of Illinois, Urbana, Illinois.*

Economic Models, E. F. Beach. New York, John Wiley and Sons, Inc., 1957. Cloth, xi+227 pp., $7.50.

In recent years there has been a great deal of work done in the applications of mathematics to economics. The result has been that there are now mathematical economists and nonmathematical economists, with little communication between the two groups. The present book is an attempt to bring these groups closer together.

The book is divided into two parts. The first part presents several of the basic models of economics, developing the mathematics needed in an informal way. The second part is quite an extensive development of the statistical con-

cepts (especially regression theory) used in testing models and a relatively short discussion of the application of these ideas to particular models.

The mathematics needed to formulate the models consists of the theory of linear equations, elementary differential equations, and elementary difference equations. These topics are developed in an uneven fashion. For example, considerable time is devoted to the concept of the derivative, and then integration is introduced as an old friend to the reader. When the author does choose to discuss a mathematical topic, he does so in a very intuitive and simple manner. The one exception to this is the discussion of the facts about the solutions of linear equations. This description is extremely vague, and the reviewer was unable to find any interpretation which would make it correct. At any rate the student is almost certain to be left with the famous false impression that for a system of equations to have a unique solution, it is necessary that there be the same number of equations as unknowns.

The author states, "Much can be learned, it is hoped, if the student has no more than some elementary algebra." This would seem to the reviewer to be very optimistic. However, a reader who has had a year of calculus and a basic statistics course should fare well. Such a reader will be helped by the fact that there is a set of exercises at the end of each chapter designed to aid not only in the understanding of the models but also in reviewing the mathematics. A complete set of answers is given. There are also a large number of suggested readings at the end of each chapter as well as many references to the literature contained in the discussion of specific models.

The presence of this book, and others like it, in the behavioral sciences is bound to have an important effect. It will show the social scientist the rewards which can come from even quite elementary mathematics. It also will force the mathematics departments to examine their introductory course and see if they are giving the kind of mathematics that students outside of the physical sciences need. Perhaps it will one day be feasible to teach the basic ideas of differential equations and difference equations in the first year of calculus. This book suggests that this might be more important to the economics student than, for example, a detailed discussion of the techniques of integration theory.

This book should be recommended not only to economics teachers and research workers but also to the mathematician who wants to see the relation of his subject to economics. The author has done an admirable job of showing the relation of the models and their consequences to economic theory without getting into technical economics. The author is to be commended on providing a book which will help to bridge not only the gap between different groups of economists but also the gap between the economist and the mathematician.—*J. Laurie Snell, Dartmouth College, Hanover, New Hampshire.*

The History of Mathematics, Joseph Ehrenfried Hofmann. New York, Philosophical Library, 1957. Cloth, xii+132 pp., $4.74.

In the Introduction to this translation of Professor Hofmann's *Geschichte der Mathematik*, Professor Hofmann refers to "the interesting and yet heretofore all too little studied development of mathematics in the Middle Ages and in the Renaissance period." The author is an authority in these areas, and it is here that the book has its greatest strength.

The book is strong in its treatment of Scholasticism and Humanism. It gives great attention to the men who translated the Greek works on mathematics, to the mathematicians of the fifteenth and sixteenth centuries, and especially to Vieta, whose work is described in considerable detail.

The History of Mathematics is so condensed that at first reading one has the impression of a catalogue of names and dates. Then, on later reading, Professor Hofmann's comments begin to emerge from these details. The comments are thought-provoking and at times controversial.

The book has suffered in translation. The language is awkward and occasionally obscure. It is likely, though this is not stated, that this translation is considerably abridged from the original text.

If this book was intended for the undergraduate student, more explanation would have been welcome. For example, the word *Contingenzwinkel* is translated "angle of contingency" where the English-American expression "angle of contact" or "horn angle" might be familiar to some, but where a description of the angle would have made the matter really clear.

The concise style makes this book difficult reading. The lack of documentation is a defect so far as the student of the history of mathematics is concerned. The mathematical layman who might well be interested in what the author notes in the Introduction as "Cursory glimpses of the inroads of mathematics into allied disciplines" will be blocked by the demands made on his knowledge of mathematics—demands that might have been eased by a bit more explanation. But the person interested in the mathematics of the thirteenth and fourteenth centuries and in the transition from the Middle Ages to modern times should not neglect this book.—*Vera Sanford, State University Teachers College, Oneonta, New York.*

Introduction to Statistical Reasoning, Philip J. McCarthy. New York, McGraw-Hill Book Company, Inc., 1957. Cloth, xiii+402 pp., $5.75.

This book is written for a one-semester, nonmathematical course in statistics. In the early chapters it considers sample and population characteristics such as location and spread. A chapter on probability is followed by a detailed discussion of the binomial and normal populations. The book concludes with the topics: elements of sampling design, chi-square procedures,

and linear association of two quantitative variables. There are exercises and references listed at the end of each chapter.

This is definitely not a reference book as it emphasizes the concepts of statistical reasoning rather than attempting to cover a large number of topics. The author's discussion of confidence intervals and statistical hypotheses is excellent. In particular, despite the nonmathematical prerequisite, the author carries through in detail the statistical inference pertaining to the binomial parameter, and he extends these techniques to the mean of a normal population. He carefully considers the power of a statistical test.

The author has exercised great care in his definitions, reasoning, and solutions. This is not always the case in some nonmathematical texts on statistics. The population characteristics have been defined in such a way as to motivate the study of calculus.

Teachers of general introductory courses in statistics should give a great deal of consideration to this text. A person lacking a calculus background who desires an introduction to statistical reasoning would do well to start with this book.—*Robert V. Hogg, University of Iowa, Iowa City, Iowa.*

Solid Geometry, Hugo Mandelbaum and Samuel Conte. New York, The Ronald Press Company, 1957. Cloth, vi+261 pp. $4.00.

The high school solid geometry course has been the subject of much discussion in recent years; and there have been many proposals for eliminating it and including the essential mensurational and spatial perception aspects in the plane geometry course. In a recent issue of this journal (November, 1956, p. 515) in an article entitled "School and College Mathematics," Professor Duren said "Solid geometry . . . was dead, and there was nothing to take its place. Even now, it will take years to construct a suitable subject to take the place of solid geometry and prepare for teaching it by mass education." If the solid geometry course is indeed to disappear, that event should be slowed by the appearance of this new text in which Professors Mandelbaum and Conte appear to have succeeded in their avowed purpose of replacing the traditional formal approach by a treatment based on concepts and principles developed by modern mathematics. Some of the concepts introduced are duality, lines and planes at infinity, projection, and invariance. A feature which will especially appeal to those teachers who have preferred to emphasize the mensurational aspect is the use of the prismoidal formula to treat volumes of various solids as special cases of a more general solid.

The text assumes one year of algebra and one year of plane geometry on the part of the student. The cosine function is introduced and used in the text. Some teachers may feel that the style of presentation in some sections requires a higher level of mathematical maturity than can be expected of high school students; but the reviewer is of the opinion that the text can be used successfully with those secondary school students who elect the solid geometry course. It also seems well suited for college courses in the subject and should be especially valuable in teacher education curricula.

The first chapter is entitled "The Elements of Solid Geometry" and the reviewer especially likes the very first section, an important section in any text. The concept of dimensional number is probably new to the student, so the reference to extension to n dimensions may well prove interesting, and it provides a good basis for review of many basic elements and ideas of plane geometry. The review is promoted by a very adequate set of questions. This chapter is also designed to help the student attain some understanding of the role of postulates in the development of a mathematical system.

The questions in the text are a definite aid to students' understanding. The problems are graded, and those starred are intended for the more ambitious students. Several problems refer to interesting and seldom recognized common applications. The sections on ruler and compass constructions of parts of three-dimensional figures aid the students' development of spatial perception but may be omitted if desired. Another excellent feature of the text is the number of calculations and constructions worked out in full. The reviewer was distressed to see that the answers to all of the questions involving calculations were included at the back of the book.

It is to be hoped that the titles of Chapters 2, 3, and 4 do not discourage high school teachers from considering this book as a possible text, since the content does not seem difficult to understand. After introducing orthogonal parallel projections, the authors develop the common solids from the projection approach: prisms and cylinders by parallel projection onto parallel planes, and pyramids and cones by central projection from one plane onto a parallel plane. A well-developed Cavalieri's Theorem approach is used for their volumes; and in Chapter 5 the formula for the volume of a conoid is developed by a similar approach, and then the general prismoidal formula is easily obtained. The latter is then used for frustrums and for the sphere and, in Chapter 6, for the spherical segment. The usual formulas are developed for zones, sectors, and lunes. The last chapter deals with some very interesting topics, including the regular polyhedra and the perspective view of three-dimensional space.

The Appendix contains tables of cosines and secants (not so called), squares and square roots, and lists of equivalents and formulas from plane and solid geometry. The facile use of the Index is hindered by the references being cited in Article numbers, since these do not appear at the tops of the pages in the text.

The text is not without a few typographical errors. The reference on page 89 to the Appendix

should be to Table 5 rather than to Table 3. On page 57 the labels of the last two points on the base line in Figure 2.26 are interchanged. However, the authors are to be commended on the clarity of the figures in the text.

The use of words in the text is generally good. However, on pages 220 and 221 "inverse" is used to describe what is conventionally called the "converse" of a theorem. The use of "axiom" to the complete exclusion of "postulate" is not in keeping with current practice. It is to be hoped that the omission of three equality signs in the last line on page 210 was unintentional. Also, the presentation of "slant angle" on page 33 seems a bit confusing upon first reading.

The reviewer is of the opinion that this text can be used satisfactorily for both high school and college courses. He does feel the authors may be a bit optimistic in their suggestion that the time required for most of the thirty-two sections will be two lessons for high school students and one for college students. The text can be adapted to particular situations by suitable omission or amplification of topics.—*John A. Schumaker, New Jersey State Teachers College, Upper Montclair, New Jersey.*

Solid Geometry, Rolland R. Smith and James F. Ulrich. New York, World Book Company, 1957. Cloth, x+266 pp., $2.88.

The simplest way to describe this text is to say that it is a revision of the well-known text *Modern Solid Geometry* by Schorling, Clark, and Smith, also published by the World Book Company. The two texts have one author in common. Comparison of the texts shows that the number of chapters and their contents are identical even to numbering of theorems. For the most part the problems are the same and the wording in the two texts is often identical, as are most of the figures.

Differences in the texts may be summarized as follows. The new text has a more pleasing format and the definitions stand out more clearly. More attention is given to the logical structure of the proofs. Some theorems are prefaced by a section, which serves as a motivation, leading up to the theorem. There are sections entitled "no proof required" distributed through the text, whose purpose is to aid the student in digesting and organizing the theorems. There are review sections on algebra at the end of each chapter.

In the reviewer's opinion, this will be a useful and popular text, probably slightly easier than its prototype.

Some comments on the standards of rigor in current texts are perhaps appropriate here. In this respect, the book under discussion is comparable to others in the field and better than many. But, like *all* solid geometry texts that have been examined by the reviewer, this one is subject to certain criticisms. These criticisms fall into three main categories: (1) omission of needed postulates; (2) inclusion of unnecessary (and perhaps false) postulates; (3) misnaming, that is, a postulate or theorem is called a definition or vice versa. In order not to make this review overly long, selected examples of these three types will be given from the present text. But it must be kept in mind that similar criticisms can be given of *all* other texts examined by the reviewer.

(1) Like all solid texts, the precise dependence on plane geometry is not fully explicit. Further, as in plane geometry, betweenness or order relations are not postulated.

(2) The postulate on page 30 is not too hard to prove. The postulates on pages 153, 164, 168, 225 are theorems and need little on limits for their proofs. It would be pedagogically better to call them theorems and then omit the proofs. The postulate on page 151 is false as is the first one on page 162.

(3) Theorems 14 and 15 constitute a definition of equality of dihedral angles.—*M. E. Shanks, Purdue University, Lafayette, Indiana.*

The following item is a selection from the world's first encyclopedia, a compendium of what men knew or believed, produced by a Spanish bishop, Isidore of Seville, in the seventh century.

"There are 22 sextarii in a bushel because God in the beginning made 22 works; there are 22 generations from Adam to Jacob; and 22 books of the Old Testament as far as Esther and 22 letters of the alphabet out of which the divine law is composed."—*Taken from* A History of Education *by Luella Cole.*

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia, and Joseph N. Payne, University of Michigan, Ann Arbor, Michigan

An approach to solving word problems

by Marilyn C. Singleton, Bexley Junior High School, Columbus, Ohio

Sometimes students become "answer conscious" when solving word problems. They feel that the main goal is to find some magic number they can call the answer. Many times the choice of the fundamental operation is a guessing game, depending to some extent on the size and position of the numbers in the problem.

Since junior high school students have a tendency to hasten through word problems without much thoughtful preparation for computation, I would like to suggest some procedures that require preparation before doing the calculations; take importance away from the final answer temporarily; give importance to planning a procedure for acquiring the answer; and give an opportunity to discover a pupil's ability to verbalize.

I have found it helpful for pupils to focus attention on the process to be used. This is done by eliminating the use of all numbers and sketching the procedure, using the appropriate phrases from the problem. Examples 1 and 2 illustrate this point:

1. Jim weighs 102 pounds and George weighs 93 pounds. What is the difference in their weights?

The student should write a description as follows:

$$\begin{array}{rl} & \text{Jim's weight} \\ (-) & \text{George's weight} \\ \hline & \text{difference in weights} \end{array}$$

2. The Smiths traveled 336 miles in one day of 8 hours' driving. What was their average speed?

$$\text{average speed} = \frac{\text{no. miles traveled}}{\text{no. hrs. driving}}$$

Students may be asked to tell what each number in a problem means before doing the calculations, as in Example 3 below:

3. Jane went to the store for her mother and bought a pound of butter for 75¢ and a loaf of bread for 23¢. She had taken a five-dollar bill with which to make the purchase. How much change should she receive?

75¢	cost of one pound of butter
23¢	cost of one loaf of bread
$5.00	amount of money Jane took
answer:	the amount of change Jane should receive

Another procedure requires pupils to write a plan for the problem but not to compute the answer, as in Example 4 below:

4. Jim charges 70¢ an hour for mowing lawns. In one week he works 2 hours

Monday night, 3 hours Tuesday, $1\frac{1}{2}$ hours Wednesday, and $2\frac{1}{2}$ hours Thursday. How much did he earn this week?

The plan would be as follows:

$$\begin{array}{rl} & 2 \\ & 3 \\ & 1\frac{1}{2} \\ (+) & 2\frac{1}{2} \\ \hline & \end{array}$$

no. of hours Jim worked

\$.70×no. hours Jim worked equals amount of money Jim earned in one week.

Although accuracy in computation is certainly important in verbal problems, the method by which a problem is attacked should become equally important. After Examples 1–4 have been done, the student, having a second look at the problems, may be better prepared to do the calculations.

These suggested procedures enable a teacher to see in what area the student is having difficulty in thinking. Such procedures also give the teacher some knowledge of the student's verbal abilities in reading and expressing thoughts in writing. With this helpful information the teacher can more effectively plan questions to bring out certain points and discover which kinds of problems need further study.

Having students write their own problems is another way to discover their powers of verbalizing. This technique also brings up subjects which are vital and more interesting to the students at the moment. This procedure could be followed by asking the students to rewrite the problem in such a form that the first answer appears in the new problem and a new question is asked that requires one of the other numbers for an answer. Example 5 is a problem posed by a student:

5. My mother baked 8 dozen cookies yesterday for a bake sale. They will sell for 40¢ per dozen. How much money will she receive for her cookies?

The "turn-about" problem, Example 6 below, would read:

6. My mother's cookies brought in \$3.20 at the bake sale yesterday. If they sold for \$.40 per dozen, how many cookies did she bake?

The two problems also point up the inverseness of multiplication and division. In almost all such problems, the second one will require the inverse operation. This provides an opportunity to show the relationship between addition and subtraction, and between multiplication and division.

I shall try to prove that exclusive preoccupation with facts will be to the detriment of science itself. Beyond that, I claim that loss of philosophical perspective endangers progress, that the need for understanding the meaning of scientific discoveries in philosophic terms is as great as the urgency to make bombs and intercontinental missiles; both are necessary, indeed equally necessary, for our cultural survival.—*Taken from Dr. Henry Margenau, "Why Teach Philosophy of Science?"* Age of Science, *Vol. 2, December,* 1956.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Yearbook Planning Committee

by Myron F. Rosskopf, Teachers College, Columbia University, Chairman

The present membership of the Yearbook Planning Committee consists of the following three persons:

Myron F. Rosskopf, Teachers College, Columbia University, Chairman (1959)

Robert E. Pingry, University of Illinois (1960)

Bruce E. Meserve, Montclair State Teachers College (1961)

The numerals following the names of the committee members represent the years their terms of office end. Appointments are made, normally, for a two-year term, beginning with the time of an annual meeting.

At the present time, three yearbooks are in preparation, each under the general chairmanship of an individual. For each of these yearbooks the chairman organized an excellent writing committee.

The tentative title of the *Twenty-fourth Yearbook* is *Concepts and Learning in Mathematics*. The chairman of the writing committee is Professor Phillip Jones, University of Michigan. During the spring of 1957, he distributed chapters to a wide group for criticism and suggestions. Every effort is being made to finish this yearbook in time for publication in 1958.

The Board of Directors approved a yearbook on arithmetic that would review, summarize, or interpret what had been done in research, in program, and in philosophy of arithmetic teaching since the *Sixteenth Yearbook*. This yearbook will be the twenty-fifth in the National Council series. Its general chairman is Professor Foster Grossnickle, Newark State Teachers College. The writing committee has been organized. Deadlines for preliminary drafts of chapters have been set so that a 1959 publication date will be achieved.

The *Twenty-sixth Yearbook*, scheduled for 1960, will be a yearbook on evaluation in mathematics. Professor Donovan Johnson, University of Minnesota, is the chairman of the writing committee. Plans have been made for the chapters, writing responsibilities have been allocated, and meetings scheduled at which progress can be discussed.

At the 1957 annual meeting in Philadelphia, the Board of Directors approved the suggestion of a yearbook whose subject would be applications of mathematics to the social sciences. Tentatively, 1961 is the year picked for the appearance of this yearbook. To date, no chairman has been appointed.

With the three yearbooks definitely under way and a fourth projected, the Yearbook Planning Committee feels that its plans are extended sufficiently far into the future. As soon as production seems to be overtaking plans, more suggestions will be submitted to the Board of Directors. But, at least for the present, there seems to be no need to make formal suggestions for yearbooks in addition to those that have been approved.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING

April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Hyman, Board of Education, Cleveland, Ohio

JOINT MEETING WITH NEA

June 30, 1958
Cleveland, Ohio
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

EIGHTEENTH SUMMER MEETING

August 19–20, 1958
Colorado State College of Education, Greeley, Colorado
Forest N. Fisch, Colorado State College of Education, Greeley, Colorado

CHRISTMAS MEETING

December 29–30, 1958
Sheraton-McAlpin Hotel, New York, New York
Abraham S. Kadish, High School of Fashion Industries, 225 West 24th Street, New York 11, New York

Other professional dates

Sectional Meetings of the Illinois Council of Teachers of Mathematics

March 15, 1958, Belleville Township High School, Belleville, Illinois
March 22, 1958, Hinsdale Township High School, Hinsdale, Illinois and Western Illinois University, Macomb, Illinois
April 16, 1958, Eastern Illinois University, Charleston, Illinois
April 19, 1958, Southern Illinois University, Carbondale, Illinois and Illinois State Normal University, Normal, Illinois
Francis R. Brown, Illinois State Normal University, Normal, Illinois

Eighth Annual Meeting of the Association of Mathematics Teachers of New York State

May 2–3, 1958
Hotel Syracuse, Syracuse, New York
Martha Neighbour, Ithaca High School, Ithaca, New York

Spring Meeting of the Arkansas Council of Teachers of Mathematics

May 3, 1958
Lafayette Hotel, Little Rock, Arkansas
Christene Poindexter, Central High School, Little Rock, Arkansas

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Letter to the editor

Dear Sir:

Each year for the past three years Central Michigan College at Mount Pleasant, Michigan has contributed from twenty-five to thirty-five student members to The National Council of Teachers of Mathematics.

Most of the students subscribe to THE MATHEMATICS TEACHER; some take *The Arithmetic Teacher;* while others subscribe to *both.* In these magazines they find articles pertinent to some of their college classes. But more important, they discover material which they can use in their student teaching and topics for discussion in their class, Materials for Teaching Mathematics in the Secondary Schools.

Somehow possession of these magazines seems to make them take on new stature. They have the air of being a little more important, they stand a bit taller, they act as though they are really ready to go forth and instruct the youth of the nation!

Perhaps I imagine all this. At any rate, I am sold on the idea of Student Members for The National Council of Teachers of Mathematics.

Gertrude V. Pratt
Central Michigan College
Mount Pleasant, Michigan

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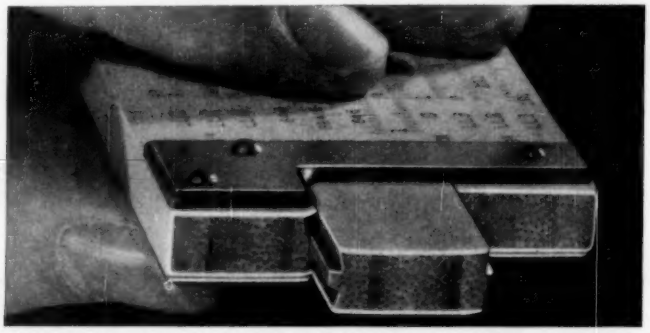
BY ROBERT JONES, *Manager of Educational Sales, Frederick Post Company*



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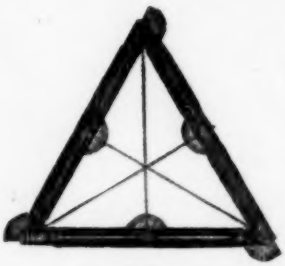

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